IESE Business School
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Working Paper
WP-915
July, 2011

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HIGHER ORDER EXPECTATIONS, ILLIQUIDITY, AND SHORT-TERM TRADING*

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Abstract

When trading frequencies between liquidity traders and short term, heterogeneously informed investors differ, asset prices reflect Higher Order Expectations (HOEs) about both fundamentals and liquidity trading, and multiple, self-fulfilling equilibria arise. Differential information and heterogeneous trading frequencies make illiquidity dependent on a coordination problem across generations of investors and generate liquidity risk. If asset prices are driven by HOEs about fundamentals, they heavily rely on public information and the market displays high illiquidity; if HOEs about fundamentals are subdued, prices rely less on public information and the market displays low illiquidity. Along the equilibrium with low illiquidity, the volume of informational trading is high, and momentum arises at short horizons. Conversely, along the equilibrium with high illiquidity the volume of informational trading is low and short term returns tend to revert. At long horizons reversal occurs.

Keywords: Expected returns, multiple equilibria, liquidity risk, average expectations, reliance on public information, momentum and reversal, Beauty Contest.

JEL Classification Numbers: G10, G12, G14

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* We thank Marcelo Fernandes, Peter Kondor, Stephen Morris, Ioanid Roşu, and seminar participants in the Econometrics Workshop at Queen Mary University of London, the IESEG workshop on Asset Pricing (IESEG, May 2010), the 2010 European Finance Association meeting (Frankfurt), the ELSE-UCL Workshop in Financial Economics (UCL, September 2010), the 2011 ESSET (Gerzensee), and the 2011 SED meeting (Gent) for their comments.

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Introduction

Liquidity plays an important role in the valuation of financial assets.[1] A relevant aspect of liquidity relates to the provision of *immediacy*, i.e. investors' readiness to hold a position in an asset in order to bridge the offsetting needs of agents who enter the market at different points in time (Grossman and Miller (1988)). A higher uncertainty over the price at which asset inventories can be unwound creates inventory risk, thereby reducing investors' ability to provide immediacy, and curtailing liquidity.[2] In markets with heterogeneous information, the risk of facing adverse selection at interim liquidation dates adds to inventory risk, potentially further increasing illiquidity. When prices reflect more efficiently the fundamentals, adverse selection is less of an issue, as the speculative opportunities offered by private information are scarcer. Price efficiency, in turn, hinges on investors' responses to their private signals. This implies that in equilibrium, both market illiquidity *and* investors' response to private information are *jointly* determined.

The impact of private information on asset prices is also at the core of the recent literature that emphasises the role of investors' Higher Order Expectations (HOEs) over asset payoffs in asset price determination (i.e., investors' expectations about other investors' expectations about ... the liquidation value). It is found that if prices are driven by HOEs over the final payoff, they heavily rely on public information, systematically departing from fundamentals compared to consensus (Allen, Morris, and Shin (2006)). While this result speaks to the informational properties of asset prices, it is less clear what its implications for market quality and asset pricing are.

In this paper we propose a theory that jointly accounts for an asset illiquidity and for the asset price potential heavy reliance on public information. We argue that in general

[1] There is by now a well established literature showing that illiquid assets command a return premium (see, e.g., Pástor and Stambaugh (2003), and Acharya and Pedersen (2005)).

[2] Several authors document intermediaries' concern over the time-varying patterns of asset illiquidity. See e.g. Comerton-Forde, Hendershott, Jones, Moulton, and Seasholes (2010).

asset prices reflect investors' HOEs about the *two* factors that influence the aggregate demand: fundamentals *and* liquidity trading. We show that it is precisely when asset prices are driven by investors' HOEs about fundamentals (and therefore heavily rely on public information) that the market displays high illiquidity, is less informationally efficient, the volume of informational trading is low, short term returns tend to revert, and expected returns are high; conversely, when HOEs about fundamentals are subdued, asset prices rely less on public information, the market hovers in a high liquidity state, is more informationally efficient, the volume of informational trading is high, returns display momentum at short horizons, and expected returns are low. At long horizons, reversal occurs.

We study a market in which overlapping generations of risk-averse, rational investors interact with liquidity traders. The former live for two periods and are informed about a pay-off that will be announced at a future date, which occurs beyond their investment horizon; the latter, instead, are uninformed and submit a random market order, potentially holding their positions during more than one trading round. When young, rational investors absorb liquidity traders' orders, thereby acting as "market-makers." Once old, the former unwind their positions against the reverting portion of the latter demand and take advantage of the new cohort of rational investors to unload the residual part of their inventory. Thus, as trading frequencies are heterogenous across investors' types, rational investors *provide* liquidity when young and *consume* it when old. This implies that when they determine their holdings, they take into account the inventory and adverse selection risk they will face at the liquidation date. We show that this feature of the model generates a number of important implications.

If in a given period rational investors anticipate that their next period peers will increase their exposure to the risky asset, they face adverse selection risk at the liquidation date which will make the market more illiquid. As a consequence, the price at which they unwind becomes more difficult to forecast with their private signal, and they scale down their response to private information. This, in turn, lowers the informativeness of equilibrium prices, opening more opportunities to speculate on private information to the informed investors who enter the market in the following period, justifying the anticipated increase in illiquidity. If, on the other hand, in a given period investors anticipate that their next period peers lower their exposure to the risky asset, the market in which they unwind will be more liquid. As a consequence, the liquidation price becomes more easy to forecast with their information, and they step up the use of private signals. This, in turn, increases the informativeness of equilibrium prices, reducing the opportunities to speculate on private information opened to the next generation of informed investors, justifying the anticipated decrease in illiquidity. We thus show that the presence of such a positive feedback loop *across trading dates* is conducive to multiple equilibria which can be ranked in terms of illiquidity.

As rational investors hold disparate signals, the illiquidity result has implications for the information aggregation properties of asset prices. Differently from the existing literature (Allen et al. (2006)), we show that in our setup a heavy reliance on public information is *closely related* to the type of equilibrium that arises. Along the equilibrium in which investors anticipate *high* future illiquidity, the risk of adverse price movements at the time of position unwinding causes strong reliance on public information; this generates a price that is systematically farther away from the liquidation value compared to consensus and poorly related to fundamentals. Conversely, when investors anticipate *low* future illiquidity, the price reacts poorly to public information, is systematically closer to the liquidation value compared to consensus, and tracks *more closely* the fundamentals. Thus, while HOEs over fundamentals are *necessary* for heavy reliance on public information, they are *not sufficient*.

We then move to relate our results to the literature on return regularities. Consistently with empirical evidence (De Bondt and Thaler (1985) and Jegadeesh and Titman (1993)), we show that at long horizons returns display reversal. However, at short horizons the correlation of returns depends on the equilibrium on which the market coordinates. Along the equilibrium with high illiquidity (and heavy reliance on public information), price adjustment to full information is staggered. This implies that unless liquidity trades display a sufficiently strong persistence (and thus exert a sufficiently continuous price pressure), short term returns are negatively autocorrelated. However, along the low illiquidity equilibrium (in which prices rely less on public information), the price closely tracks the full information value from the first trading round. Hence, in the remaining rounds of trade prices adjust little, implying that returns are positively correlated, and momentum occurs. This provides a novel explanation for return predictability which departs from the behavioral finance paradigm.[3]

Our results also have implications for the informational content of volume. Indeed, we show that the expected volume of informational trading is high (low) along the equilibrium with low (high) illiquidity. This, coupled with the behavior displayed by short term returns, implies that in our context a high volume due to informed trading predicts return continuation, consistently with Llorente, Michaely, Saar, and Wang (2002). Note, however, that as in our setup momentum can also occur along the high illiquidity equilibrium (provided a sufficiently strong persistence in liquidity trading), our model also implies that this phenomenon can either signal rapid convergence to the full information value, or the existence of a continuing price pressure due to liquidity trading. In this respect, the volume of informational trading is key to distinguish between the two explanations.

This paper is related to a growing literature that points out the relevance of higher order expectations in influencing asset prices. As our previous discussion suggests, we depart from the main tenet of this literature and point out that the heavy reliance of

[3]We review the relationship between our results and those of the literature on behavioral finance in Section 6.

prices on public information is closely related to the transience of liquidity traders' demand.[4] Bacchetta and van Wincoop (2008) study the role of higher order beliefs in asset prices in an infinite horizon model showing that higher order expectations add an additional term to the traditional asset pricing equation, the higher order "wedge," which captures the discrepancy between the price of the asset and the average expectations of the fundamentals. Kondor (2009), in a model with short-term Bayesian traders, shows that public announcements may increase disagreement, generating high trading volume in equilibrium. Nimark (2007), in the context of Singleton (1987)'s model, shows that under some conditions both the variance and the impact that expectations have on the price decrease as the order of expectations increases. Other authors have analyzed the role of higher order expectations in models where traders hold different initial beliefs about the liquidation value. Biais and Bossaerts (1998) show that departures from the common prior assumption rationalise peculiar trading patterns whereby traders with low private valuations may decide to buy an asset from traders with higher private valuations in the hope to resell it later on during the trading day at an even higher price. Cao and Ou-Yang (2005) study conditions for the existence of bubbles and panics in a model where traders' opinions about the liquidation value differ.[5] Banerjee, Kaniel, and Kremer (2009) show that in a model with heterogeneous priors, differences in higher order beliefs may induce price drift. Ottaviani and Sørensen (2009), in a static model of a binary prediction market where agents hold heterogeneous prior beliefs and are wealth constrained (either exogenously, by the rules of the market, or because of their attitude towards risk), show that the fully revealing REE price underweights aggregate private information.

The paper is also related to the literature that investigates the relationship between the impact of short-term information on prices and investors' reaction to their private

[4]In a related paper, we show that a similar conclusion holds in a model with long term investors (see Cespa and Vives (2011)).

[5]Kandel and Pearson (1995) provide empirical evidence supporting the non-common prior assumption.

signals (see, e.g. Brown and Jennings (1989), Vives (1995), Cespa (2002), Albagli (2011) and Vives (2008) for a survey). Several authors have argued that when private information is related to an event which occurs beyond the date at which investors liquidate their positions, the latter act on their signals only if they expect them to be reflected in the price at which they liquidate (see, e.g., Dow and Gorton (1994) and Froot, Scharfstein, and Stein (1992)). In our context the main driver in investors' reaction to private signals is the uncertainty over the illiquidity of the market in which they unwind their positions. Indeed, the impossibility to pin down a particular level of future illiquidity is responsible for investors' over- or under-reaction to private information which in turn feeds back into different equilibrium levels of illiquidity.

Finally, our paper is also related to the literature on behavioral finance (see Barberis and Thaler (2003) for a survey). As argued above, our setup can explain return regularities, emphasizing the predictive role of the volume of informational trading in a model where all investors, except for liquidity traders, are free from behavioral biases.

The rest of the paper is organized as follows: in the next section we review the related literature. Next, we spell out the model's assumptions and analyse the model in a setting with homogeneous information. We then turn to the market with heterogeneous information. In the following section we relate our illiquidity result to the literature on HOEs. We then draw the implications of our analysis for the literature on return regularities and volume. In the final section we summarize our results discuss their empirical implications. Most of the proofs are relegated to the appendix.

1 A three-period market with short term investors

Consider a dynamic version of the noisy rational expectations market analyzed by Admati (1985), where a single risky asset with liquidation value $v \sim N(\bar{v}, \tau_v^{-1})$, and a riskless asset with unitary return are traded by a continuum of risk-averse speculators in the interval $[0, 1]$ together with liquidity traders for $N = 3$ periods. At any trading date n, a cohort of of risk averse, rational investors in the interval $[0, 1]$ enters the market, loads a position in the risky asset which it unwinds in period $n + 1$. A rational investor i has CARA preferences (denote with γ the common risk-tolerance coefficient) and maximizes the expected utility of his short term profit $\pi_{in} = (p_{n+1} - p_n)x_{in}$.[6] The short term horizons of rational investors can be justified on grounds of liquidity needs or incentive reasons related to performance evaluation, or because of difficulties associated with financing long-term investment in the presence of capital market imperfections (see Holmström and Ricart i Costa (1986), and Shleifer and Vishny (1990)). An investor i who enters the market in period n receives a signal $s_{in} = v + \epsilon_{in}$, where $\epsilon_{in} \sim N(0, \tau_\epsilon^{-1})$, v and ϵ_{in} are independent for all i, n and error terms are also independent both across time periods and investors. We assume that informed investors in period n observe the past period prices up to period $n - 1$, denoted by $p^{n-1} \equiv \{p_t\}_{t=1}^{n-1}$, and submit a linear demand schedule (generalized limit order) to the market $X_n(s_{in}, p^{n-1}, p_n) = a_n s_{in} - \varphi_n(p^n)$ indicating the desired position in the risky asset for each realization of the equilibrium price p_n. The constant a_n denotes the private signal responsiveness, while $\varphi_n(\cdot)$ is a linear function of the equilibrium prices p^n.

The stock of liquidity trades is assumed to follow an AR(1) process:

$$\theta_n = \beta\theta_{n-1} + u_n, \tag{1}$$

where $\beta \in [0, 1]$ and $\{u_n\}_{n=1}^N$ is an i.i.d. normally distributed random process (independent of all other random variables in the model) with $u_n \sim N(0, \tau_u^{-1})$. If $\beta = 1$, $\{\theta_n\}$ follows a random walk and we are in the usual case of independent liquidity trade increments $u_n = \theta_n - \theta_{n-1}$ (e.g., Kyle (1985), Vives (1995)). If $\beta = 0$, then liquidity trading

[6] We assume, without loss of generality, that the non-random endowment of investors is zero.

is i.i.d. across periods (this is the case considered by Allen et al. (2006)).

This random process could be interpreted in the following way. Suppose $\beta < 1$, then at $n > 1$ four groups of agents are in the market: the $n-1$-th and n-th generations of rational investors with demands $x_{n-1} \equiv \int_0^1 x_{in-1}di$, and $x_n \equiv \int_0^1 x_{in}di$, a fraction $1-\beta$ of the $n-1$-th generation of liquidity traders who revert their positions θ_{n-1}, and the new generation of liquidity traders with demand u_n. Considering the first two trading dates and letting $\Delta x_2 \equiv x_2 - x_1$, $\Delta\theta_2 \equiv \theta_2 - \theta_1 = u_2 + (\beta - 1)\theta_1$, at equilibrium this implies

$$x_1 + \theta_1 = 0$$

$$x_2 - x_1 + u_2 - (1-\beta)\theta_1 = 0 \Leftrightarrow \Delta x_2 + \Delta\theta_2 = 0 \Leftrightarrow x_2 + \beta\theta_1 + u_2 = 0.$$

At date 1 the first cohort of rational investors clears the share supply θ_1. At date 2 a fraction $(1-\beta)\theta_1$ of the trades initiated by liquidity traders at time 1 reverts. Hence, period 1 rational investors clear the complementary fraction $\beta\theta_1 = -\beta x_1$ against the new aggregate demand: $x_2 + u_2$. In general, the lower is β, the higher is the fraction of period n liquidity traders who revert their positions at time $n+1$, and the lower is the fraction of rational investors' n-th period position that is cleared against the $n+1$-th aggregate demand.[7]

Besides capturing an empirically documented feature of the demand of liquidity traders (see, e.g., Easley et al. (2008)), assuming persistence in liquidity trades allows to model in a parsimonious way the possibility that agents in the market have different horizons: when $\beta = 0$ each generation of rational investors and liquidity traders have the same investment horizon; as β grows, investment horizons become increasingly different. Table 1 displays the evolution of liquidity trades and rational investors' positions in the three periods.

[7] The AR(1) assumption for liquidity traders' demand is not new in the literature. For instance, He and Wang (1995) and Cespa and Vives (2011) consider a model with long term investors in which liquidity

Trading Date			1	2	3
Liquidity traders	Position at $n-1$	Reverts	$-$	$(1-\beta)\theta_1$	$(1-\beta)(\beta\theta_1+u_2)$
		Holds	$-$	$\beta\theta_1$	$\beta(\beta\theta_1+u_2)$
	New shock		u_1	u_2	u_3
	Position at n		$\theta_1=u_1$	$\theta_2=\beta\theta_1+u_2$	$\theta_3=\beta(\beta\theta_1+u_2)+u_3$
Rational investors	Position at n		x_1	x_2	x_3
	Reverting		$-$	x_1	x_2

Table 1: The evolution of liquidity trades and rational investors' positions in the three periods. The position of liquidity traders in period n is given by the sum of "Holds" and "New shock." Market clearing at n requires that the sum of liquidity traders' and rational investors' positions offset each other.

We denote by $E_{in}[Y]=E[Y|s_{in},p^n]$, $E_n[Y]=E[Y|p^n]$ ($\text{Var}_{in}[Y]=\text{Var}[Y|s_{in},p^n]$, $\text{Var}_n[Y]=\text{Var}[Y|p^n]$), respectively the expectation (variance) of the random variable Y formed by a trader conditioning on the private and public information he has at time n, and that obtained conditioning on public information only. The consensus opinion about the fundamentals at time n is denoted by $\bar{E}_n[v]\equiv\int_0^1 E_{in}[v]di=\int_0^1 E_{in}[v]di$. Finally, we let $\alpha_{E_n}=\tau_\epsilon/\tau_{in}$, where $\tau_{in}\equiv(\text{Var}_{in}[v])^{-1}$ and make the convention that, given v, at time n the average signal $\int_0^1 s_{in}di$ equals v almost surely (i.e. errors cancel out in the aggregate: $\int_0^1\epsilon_{in}di=0$). Therefore, we have $E_{in}[v]=\alpha_{E_n}s_{in}+(1-\alpha_{E_n})E_n[v]$, and $\bar{E}_n[v]=\alpha_{E_n}v+(1-\alpha_{E_n})E_n[v]$.

2 The market with homogeneous information

In this section we assume away private information (i.e., we impose $\tau_{\epsilon_n}=0$). In this case, rational investors can perfectly observe the stock of liquidity trades and always act as market makers, providing immediacy as in Grossman and Miller (1988). It is then

trading is generated by an AR(1) process.

possible to show that

Proposition 1. *In the market without private information, there exists a unique equilibrium in linear strategies where for* $n = 1, 2, 3$*:*

$$X_n(\theta_n) = -\theta_n \tag{2}$$

$$p_n = \bar{v} + \Lambda_n \theta_n, \tag{3}$$

and for $n = 1, 2$*:*

$$\Lambda_n = \frac{\mathrm{Var}_n[p_{n+1}]}{\gamma} + \beta \Lambda_{n+1}, \tag{4}$$

while $\Lambda_3 = (\gamma \tau_v)^{-1}$.

According to (2) and (3), rational investors always take the other side of the order flow, buying the asset at a discount when $\theta_n < 0$, and selling it at a premium otherwise. For a given realization of the innovation in liquidity traders' demand u_n, the larger is Λ_n, the larger is the adjustment in the price rational investors require in order to absorb it. Therefore, Λ_n proxies for the *illiquidity* of the market. According to (4), at any $n < 3$, illiquidity captures two effects, both of which are related to inventory risk. On the one hand, due to the randomness of liquidity trades, investors bear the risk related to the price at which they will unwind their position; on the other hand, due to the persistence of liquidity traders' demand, they anticipate the impact that the unwinding of a fraction of their holdings to the next cohort of rational investors will have on the next period price as a result of the *increased* risk exposure this implies for the next cohort of rational investors. The former effect is reflected in the conditional variance of the liquidation price (scaled by risk tolerance) $\mathrm{Var}_n[p_{n+1}]/\gamma$; the latter, is instead captured by the anticipated illiquidity component $\beta\Lambda_{n+1}$. Recursively substituting in (4) implies that

$$\begin{aligned}\Lambda_n &= \frac{\mathrm{Var}_n[p_{n+1}]}{\gamma} + \beta\Lambda_{n+1} \\ &= \frac{\mathrm{Var}_n[p_{n+1}]}{\gamma} + \beta\left(\frac{\mathrm{Var}_{n+1}[p_{n+2}]}{\gamma} + \beta\Lambda_{n+2}\right) \\ &= \cdots \\ &= \sum_{t=n}^{N} \beta^{t-n}\frac{\mathrm{Var}_t[p_{t+1}]}{\gamma}, \end{aligned} \tag{5}$$

and substituting the latter in (3) and rearranging yields

$$p_n = \bar{v} + \left(\sum_{t=n}^{N} \beta^{t-n}\frac{\mathrm{Var}_t[p_{t+1}]}{\gamma}\right)\theta_n. \tag{6}$$

The expression in (6) is reminiscent of Amihud and Mendelson (1986) who in a OLG-model with exogenous illiquidity costs show that the impact of illiquidity on asset prices is captured by the present value of the stream of transaction costs through the life of the asset. In our context, illiquidity is endogenous, and reflects the (risk-tolerance weighted) uncertainty over the price at which investors unwind their positions. As the stock of liquidity trading displays persistence, θ_n affects the price at $t = n, n+1, \ldots, N$ with a decreasing impact captured by β^{t-n}. Hence, expression (6) shows that due to liquidity trading persistence, the n-th price reflects the (risk-tolerance weighted) uncertainty over the liquidation price faced by the $t = n, n+1, \ldots, N$ generation of rational investors weighted by the persistence parameter.

A higher liquidation price risk increases illiquidity and investors' expected returns (conditional on θ_n):

$$
\begin{aligned}
E_n[p_{n+1} - p_n] &= E_n\left[-\frac{\text{Var}_n[p_{n+1}]}{\gamma}\theta_n + \Lambda_{n+1}u_{n+1}\right] \\
&= -\frac{\text{Var}_n[p_{n+1}]}{\gamma}\theta_n. \qquad (7)
\end{aligned}
$$

An increase in liquidity traders' persistence has two consequences: on the one hand, it implies a stronger effect due to investors' position unwinding. This, in turn, pushes investors to adjust more harshly the price for a given realization of liquidity traders' demand (see equation (6)).[8] On the other hand, it augments the persistence of price pressure across trading dates, implying that at short horizons returns can display continuation.

Corollary 1. *In the market with homogeneous information:*

1. $\partial\Lambda_n/\partial\beta > 0$, *for* $n = 1, 2$.
2. *If* $\gamma^2\tau_u\tau_v < 2/(\sqrt{5} - 1)$, *for* β *sufficiently high, and* $n \geq 3$, $\text{Cov}[p_n - p_{n-1}, p_{n-1} - p_{n-2}] > 0$.
3. *For* $\beta \in (0, 1]$, $\text{Cov}[v - p_3, p_1 - \bar{v}] < 0$. *For* $\beta = 0$, $\text{Cov}[v - p_3, p_1 - \bar{v}] = 0$.

With no private information, when investors face considerable uncertainty over the value of their position (due either to a low unconditional precision on the fundamentals τ_v, or to a low precision in the demand of liquidity traders τ_u), and such uncertainty sizeably reduces their expected utility (due to γ low), the price adjustment needed to absorb the position of liquidity traders is large. When, in addition, liquidity trading is sufficiently persistent, the risk associated with a given position held by rational investors propagates across trading dates. As a consequence, a given initial adjustment is less likely to revert, yielding momentum at short horizons. Furthermore, as with $\beta > 0$ the

[8] Equivalently, all else equal, a larger β by making the demand of liquidity traders more persistent increases the risk that investors at n shed on the shoulders of their next period peers who in turn require a larger compensation which feeds back into the illiquidity of the period n market.

risk associated with the initial liquidity shock θ_1 propagates until the last trading date, persistence in liquidity trading also yields reversal at long horizons. At date 4, in fact, the asset is liquidated and θ_1 has an opposite effect on the first and third period prices.

3 The market with heterogeneous information

With heterogeneous information, the aggregate demand is driven by liquidity trading and fundamental information shocks. This creates a signal extraction problem for investors and affects their strategies. As a consequence, illiquidity proxies for the two risks investors face: inventory and adverse selection. As we will argue, this can generate multiple, self-fulfilling equilibria. We start by giving a general characterisation of the equilibrium. The following proposition characterises equilibrium prices:

Proposition 2. *At any linear equilibrium of the market with heterogeneous information the price is given by*

$$p_n = \alpha_{P_n}\left(v + \frac{\theta_n}{a_n}\right) + (1 - \alpha_{P_n})E_n[v], \tag{8}$$

where $\theta_n = u_n + \beta\theta_{n-1}$*, and* a_n*,* α_{P_n} *denote respectively the responsiveness to private information at time* n *displayed by investors and by the equilibrium price.*

According to (8), at period n the equilibrium price is a weighted average of the market expectation about the fundamentals v, and a monotone transformation of the n-th period aggregate demand intercept.[9] A straightforward rearrangement of (8) yields

$$\begin{aligned} p_n - E_n[v] &= \frac{\alpha_{P_n}}{a_n}\left(a_n\left(v - E_n[v]\right) + \theta_n\right) \qquad (9) \\ &= \Lambda_n E_n[\theta_n], \end{aligned}$$

[9]This is immediate since in any linear equilibrium $\int_0^1 x_{in} di + \theta_n = a_n v + \theta_n - \varphi_n(p^n)$.

where $\Lambda_n \equiv \alpha_{P_n}/a_n$, implying that there is a discrepancy between p_n and $E_n[v]$ which, as in (3), captures a premium which is proportional to the expected stock of liquidity that is available at time n. Given that via the observation of the aggregate demand, investors infer fundamentals information this premium drives a wedge between the equilibrium price p_n and the semi-strong efficient price $E_n[v]$:

Corollary 2. *At any linear equilibrium, the price incorporates a premium above the semi-strong efficient price:*

$$p_n = E_n[v] + \Lambda_n E_n[\theta_n], \tag{10}$$

where for $n = 1, 2$,

$$\Lambda_n = \frac{\text{Var}_{in}[p_{n+1}]}{\gamma} + \beta\Lambda_{n+1}, \tag{11}$$

while $\Lambda_3 = 1/(\gamma\tau_{i3})$.

Expressions (10) and (11) parallel (3) and (4), in that also in the presence of heterogeneous information rational investors require a compensation to clear the market. However, in this case due to the presence of informed investors, price changes also reflect the arrival of new information about the asset liquidation value, and Λ_n only captures one component of the illiquidity at time n. More formally, denoting $\Delta a_n = a_n - \beta a_{n-1}$, and by $z_n = \Delta a_n v + u_n$ the "new" information reflected in the n-th period aggregate demand by the change in position of informed investors and the new liquidity shock, we obtain

Corollary 3. *At any linear equilibrium, short term returns are given by*

$$p_n - p_{n-1} = \lambda_n \left(z_n + \Delta a_n \frac{\alpha_{P_{n-1}} - \alpha_{E_{n-1}}}{a_{n-1}} E_{n-1}[\theta_{n-1}] - \Delta a_n p_{n-1} \right), \tag{12}$$

where

$$\lambda_n = \Lambda_n + (1 - \Lambda_n a_n)\frac{\Delta a_n \tau_u}{\tau_n}, \tag{13}$$

captures the illiquidity of the market at time n.

The expression in (13) captures the illiquidity of the market and corresponds to the price impact due to the innovation in liquidity traders' demand ($\lambda_n \equiv \partial p_n/\partial u_n$). According to (13), illiquidity reflects the "total" price impact of net trades and is given by the sum of two components: the first component (Λ_n) corresponds to the illiquidity measure in the market with no private information, and reflects the inventory risk investors bear when clearing the position of liquidity traders accounting for the impact that the period n expected liquidity shock has on the aggregate demand at $n, n+1, \ldots, N$. Indeed, in view of (11) in this case too we can obtain expressions similar to (5) and (6):

$$\Lambda_n = \sum_{t=n}^{N} \beta^{t-n} \frac{\mathrm{Var}_{it}[p_{t+1}]}{\gamma} \tag{14}$$

$$p_n = E_n[v] + \left(\sum_{t=n}^{N} \beta^{t-n} \frac{\mathrm{Var}_{it}[p_{t+1}]}{\gamma}\right) E_n[\theta_n]. \tag{15}$$

The second component ($(1 - \Lambda_n a_n)\Delta a_n \tau_u/\tau_n$) reflects the adverse selection risk investors face owing to the presence of heterogeneous information.[10] Note that if $\beta > 0$, adverse selection risk can either magnify or reduce illiquidity, depending on the sign of Δa_n. Intuitively, when $\beta > 0$ informed investors in period $n-1$ unwind a fraction of their orders against the new cohort of investors who enter the market in the following period. How informed investors in period n decide to react to these orders depends on the speculative opportunities that they envisage to exploit. If, given the information that has been revealed in the previous trading rounds, period n informed investors anticipate the

[10] Indeed, $\Delta a_t \tau_u/\tau_t$ denotes the OLS regression coefficient assigned to z_t in the regression of v over $\{z_1, z_2, \ldots, z_n\}$.

possibility to exploit their private information, they absorb these orders, increasing their exposure to the risky asset. In this case $a_n > \beta a_{n-1}$ and thus $\Delta a_n > 0$ and, as more private information is reflected in the price in period n, adverse selection has a positive effect on illiquidity. However, if in period n informed investors view little speculative opportunities, they choose to lower their exposure to the risky asset. In this case $a_n < \beta a_{n-1}$, $\Delta a_n < 0$, and the adverse selection component has a negative impact on (i.e., it lowers) illiquidity.[11]

An immediate consequence of (12) is that

Corollary 4. *At any linear equilibrium*

$$E_n[p_{n+1} - p_n] = -\frac{\text{Var}_{in}[p_{n+1}]}{\gamma} E_n[\theta_n]. \tag{16}$$

Expression (16) has two implications: on the one hand, it shows that short term returns are predictable based on *public* information and that such predictability depends on the possibility to correctly extrapolate liquidity traders' demand from the aggregate demand; on the other hand, similarly to the market with no information (see equation (7)), for a given expected demand of liquidity traders in period n, higher price risk commands higher expected returns.[12] Due to the presence of informed trading, however, an estimated positive (negative) liquidity shock, does not necessarily lead investors to take the other side of the market:

Corollary 5. *At any linear equilibrium, a rational investor's strategy is given by*

$$X_n(s_{in}, z^n) = \frac{a_n}{\alpha_{E_n}}(E_{in}[v] - p_n) + \frac{\alpha_{P_n} - \alpha_{E_n}}{\alpha_{E_n}} E_n[\theta_n], \tag{17}$$

[11]In this discussion we are taking $\Lambda_n a_n < 1$, which, as we will argue in the next section is always true in the equilibrium of the 2-period market.

[12]If prices were set by risk neutral market makers (as, e.g., in Vives (1995)) the marginal investor would not bear price risk, the market would be semi-strong efficient, and expected returns would be unpredictable based on public information.

where $z^n = \{z_t\}_{t=1}^n$, $\alpha_{E_n} = \tau_\epsilon/\tau_{in}$,

$$\alpha_{P_n} = \alpha_{E_n}\left(1 + \gamma\tau_n \sum_{t=n+1}^{N} \Lambda_t \left(\beta\rho_{t-1} - \rho_t\right)\right),\ n = 1, 2, \tag{18}$$

$\alpha_{P_3} = \alpha_{E_3}$, $\rho_n = a_n/(\gamma\tau_\epsilon)$, $a_3 = \gamma\tau_\epsilon$, *and for* $n = 1, 2$,

$$a_n = \gamma\frac{\lambda_{n+1}\Delta a_{n+1}}{\text{Var}_{in}[p_{n+1}]}\alpha_{E_n}. \tag{19}$$

According to (17), at period $n < 3$, a rational investor's strategy is the sum of two components. The first component captures the investor's activity based on his *private* estimation of the difference between the fundamentals and the n-th period equilibrium price. This is akin to "long-term" speculative trading, aimed at taking advantage of the investor's superior information on the liquidation value of the asset. The second component captures the investor's activity based on the extraction of order flow, i.e. *public*, information. This trading is instead aimed at timing the market by exploiting short-run movements in the asset price determined by the evolution of the future aggregate demand. Upon observing this information, and depending on the sign of the difference $\alpha_{P_n} - \alpha_{E_n}$, rational investors engage either in "market making" (when $\alpha_{P_n} - \alpha_{E_n} < 0$, thereby accommodating the aggregate demand) or in "trend chasing" (when $\alpha_{P_n} - \alpha_{E_n} > 0$, thus following the market). To fix ideas, suppose that $E_n[\theta_n] > 0$ (which, given (16), implies $E_n[p_{n+1} - p_n] < 0$). Given that

$$E_n[\theta_n] = a_n\left(v - E_n[v]\right) + \theta_n,$$

rational investors' reaction to this observation depends on whether they believe it to be driven by liquidity trades or fundamentals information. In the former case, they anticipate that the impact of their position unwinding on the $n+1$ price will be negative.

Hence, if $\alpha_{P_n} < \alpha_{E_n}$, they take the other side of the market, acting as market makers and shorting the asset (at a premium above $E_n[v]$) in the expectation of buying it back at a lower price in period $n+1$. If, on the other hand, they attribute their estimate to fundamentals information, they instead anticipate that their position unwinding will have a positive impact on the $n+1$ price. Hence, if $\alpha_{P_n} > \alpha_{E_n}$, they buy the asset (once again at a premium above $E_n[v]$), expecting to resell it at a price that reflects the positive news, effectively chasing the trend.

Accordingly, the impact that rational investors' estimate of the supply shock has on p_n and on $p_{n+1} - p_n$ changes depending on whether they act as contrarians or trend chasers. To see this it suffices to impose market clearing on (17) and solve for the equilibrium price, obtaining:

$$\int_0^1 X_n(s_{in}, z^n)di + \theta_n = 0 \Leftrightarrow p_n = \bar{E}_n[v] + \frac{\alpha_{P_n} - \alpha_{E_n}}{a_n} E_n[\theta_n] + \frac{\alpha_{E_n}}{a_n}\theta_n, \tag{20}$$

where $\bar{E}_n[v] \equiv \int_0^1 E_{in}[v]di = \alpha_{E_n} v + (1 - \alpha_{E_n})E_n[v]$, and $\alpha_{E_n} = \tau_\epsilon/\tau_{in}$. Shifting the time index one period ahead in (12), we obtain:

$$p_{n+1} - p_n = \lambda_{n+1}\left(z_{n+1} + \Delta a_{n+1}\frac{\alpha_{P_n} - \alpha_{E_n}}{a_n} E_n[\theta_n] - \Delta a_{n+1} p_n\right). \tag{21}$$

According to (20) and (21) when rational investors act as contrarians ($\alpha_{P_n} < \alpha_{E_n}$), an estimated positive supply shock at time n ($E_n[\theta_n] > 0$) has a negative impact on p_n and on $p_{n+1} - p_n$; conversely, when they chase the market ($\alpha_{P_n} > \alpha_{E_n}$), the same estimate has a positive impact both on p_n and on $p_{n+1} - p_n$. This latter possibility can never occur in a market without private information. Indeed, as noted in the previous section, in such a market prices are not tied to a persistent factor, and thus are *never* expected to trend.

4 Multiple equilibria and illiquidity

In this section we restrict attention to the case $N = 2$. This allows us to prove existence and analytically characterize the set of equilibrium solutions:

Proposition 3. *When $N = 2$, linear equilibria always exist. If $\beta \in (0, 1]$:*

1. *There are two equilibria in which the responsiveness to private signals are $a_2 = \gamma\tau_\epsilon$, and a_1^*, a_1^{**} where $a_1^* < a_1^{**}$ (see (A.44), and (A.45), in the appendix for explicit expressions). When $a_1 = a_1^*$ investors are contrarians ($\alpha_{P_1} < \alpha_{E_1}$), while when $a_1 = a_1^{**}$, they chase the market ($\alpha_{P_1} > \alpha_{E_1}$);*

2. *When $a_1 = a_1^*$, $\lambda_2(a_1^*) > 0$, while when $a_1 = a_1^{**}$, $\lambda_2(a_1^{**}) < 0$. Furthermore, $|\lambda_2(a_1^{**})| < \lambda_2(a_1^*)$, and prices are more informative along the equilibrium with low second period illiquidity.*

If $\beta = 0$, the equilibrium is unique: $a_2 = \gamma\tau_\epsilon$,

$$a_1 = \lim_{\beta \to 0} a_1^* = \frac{\gamma a_2^2 \tau_u}{1 + \gamma a_2 \tau_u}, \tag{22}$$

and investors are contrarians ($\alpha_{P_1} < \alpha_{E_1}$).

According to the above result, multiple equilibria where investors display different levels of private signal responsiveness a_1 can arise. The intuition is as follows. In the first period, investors use their private signals to forecast the price at which they unwind their position, p_2. However, with heterogeneous information, prices are driven by fundamentals information, and liquidity trades. The less illiquid is the second period market (i.e., the lower is λ_2), the weaker is the reaction of p_2 to the information contained in the second period aggregate demand, and the easier it is for informed investors in the first period to predict the second period price with their signals. When $\beta > 0$, if first period investors anticipate that their second period peers will increase their exposure to the risky asset (i.e. that they will unwind in the hands of informed investors), this implies that they will suffer from adverse selection risk at the liquidation date. Thus, they expect the market in period 2 to be more illiquid. As a consequence, they scale down their response to private information, conveying less information to the price and opening more speculative opportunities to second period investors. In this case, indeed, the second period market is more illiquid. If, on the other hand, first period investors anticipate that investors in the second period reduce their exposure to the risky asset, they expect a more liquid market in period 2. As a consequence, they ramp up their response to their private signals, conveying more information to the price and narrowing the speculative opportunities available to second period investors. In this case, thus, the market in the second period is less illiquid (see Figure 1).

When $\beta = 0$, first period investors anticipate unwinding their position against the reverting demand of liquidity traders. In this case, the adverse selection component of the second period price impact is always positive, implying that prices react more aggressively to z_2. As a consequence, first period investors scale back their response to private information.

The existence of a negative (and small) price impact of trades, along the low illiquidity equilibrium, is consistent with Boehmer and Wu (2006) who find a negative association between "uninformed" investors' imbalances and contemporaneous returns.[13] Saar and Linnainmaa (2010) in their analysis of brokers' activity in the Helsinki Stock Exchange, also find that price impacts of households (who are arguably uninformed) are negative. In our context, in the low illiquidity equilibrium, the price impact is small and negative exactly because, in view of a very informative first period price, second period

[13]In Boehmer and Wu (2006), uninformed investors are individuals, market makers, and institutions who adopt program trades.

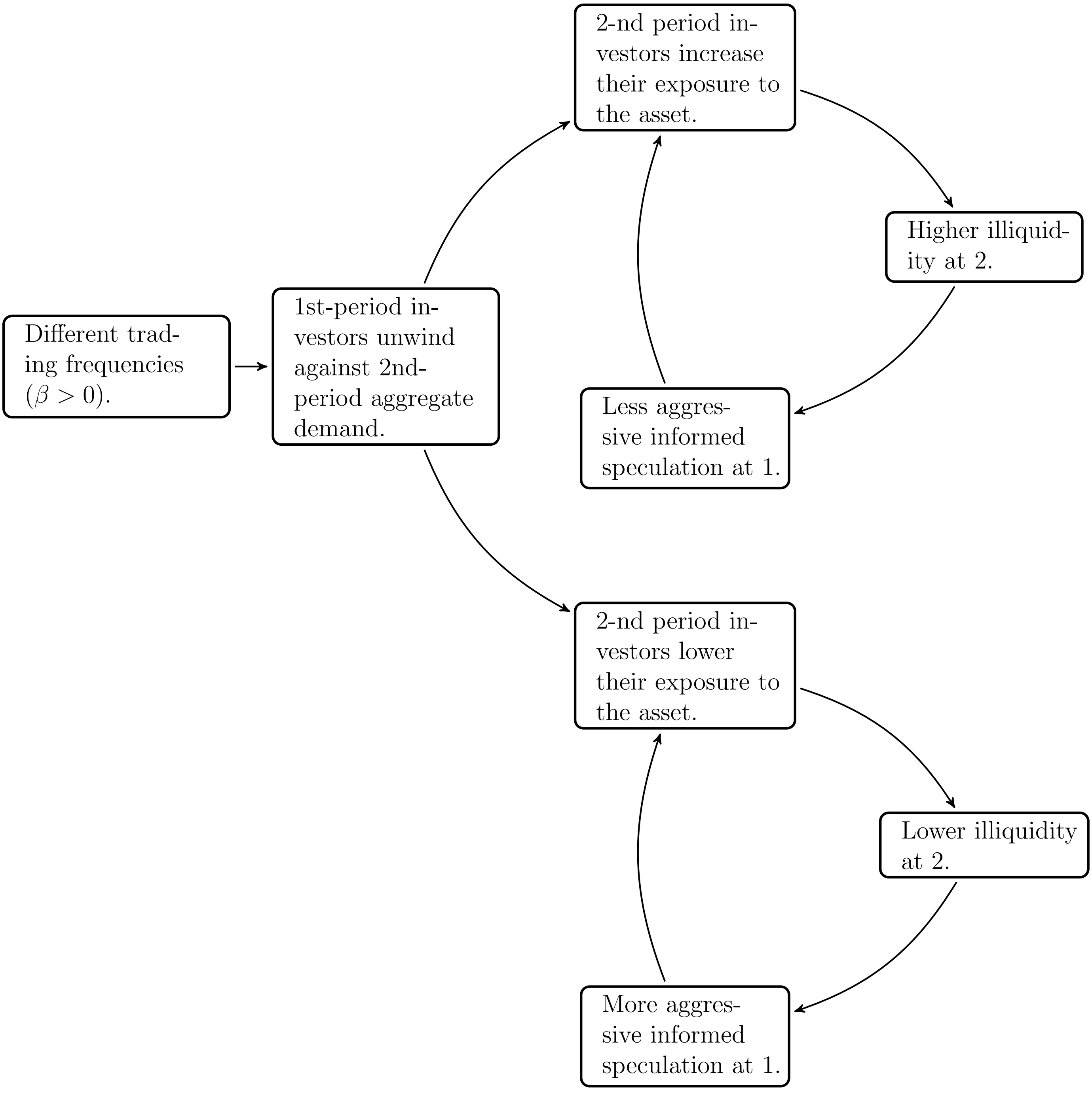

Figure 1: Multiple equilibria with self-fulfilling illiquidity.

informed investors reduce their exposure to the risky asset, implying that liquidity (i.e., uninformed) traders end up having a more relevant role in clearing the market.

Proposition 3 clarifies that the responsiveness to private information *and* the reaction to the estimated demand of liquidity traders (measured by $(\alpha_{P_1} - \alpha_{E_1})/\alpha_{E_1}$, see (17)) are closely related. Indeed, the stronger is investors' reaction to private signals, the more likely that the latter estimate of the liquidity stock is influenced by fundamentals information, implying that investors chase the trend. This is what happens in the equilibrium with low second period illiquidity. Conversely, in the equilibrium along which second period illiquidity is high, investors scale back the responsiveness to private signals, implying that estimates of liquidity trades are more likely to signal non-fundamentals driven orders. This justifies contrarian behavior.

Other authors have argued that when private information is related to an event which occurs beyond the date at which investors liquidate their positions, the latter act on their signals only if they expect them to be reflected in the price at which they liquidate. This effect is responsible for Dow and Gorton (1994)'s arbitrage chains, Froot, Sharfstein and Stein (1992)'s herding on short term private information, as well as the attenuated response to private information by short term traders in Vives (1995). In the present context a similar effect is at work. Note, however, that the main driver in first period investors' reaction to private signals *is not* the anticipation of a *strong* impact of private information on the liquidation price. Indeed, if that happened the second period market would not necessarily be liquid. It is rather the anticipation of a lower informational advantage held by second period investors, which implies a lower illiquidity for first period investors when unwinding their positions that matters.

As the persistence in liquidity trading is reduced, in both equilibria first period informed investors speculate more aggressively on their private information:

Corollary 6. *When $N = 2$, and $\beta \in (0, 1)$, at any linear equilibrium $\partial a_1/\partial\beta < 0$.*

Proof. When $N = 2$, rearranging (19) yields

$$\phi(a_1) \equiv \lambda_2 \tau_{i2} a_1 - \gamma \Delta a_2 \tau_u \tau_\epsilon = 0.$$

The result follows immediately, since from implicit differentiation of the above with respect to β:

$$\frac{\partial a_1}{\partial \beta} = -\frac{\gamma \tau_u a_1 (a_2 - a_1)}{(1 + \gamma \tau_u \Delta a_2) + \gamma \beta \tau_u (a_2 - a_1)} < 0,$$

independently of the equilibrium that arises; in the high illiquidity equilibrium we have $\beta a_1 < a_1 < a_2 \equiv \gamma \tau_\epsilon$, and in the low illiquidity equilibrium $a_1 > a_2/\beta > a_2$ and $1 + \gamma \tau_u \Delta a_2 < 0$. □

The intuition for this result is as follows: along the equilibrium with high illiquidity, a higher β implies that a larger fraction of the position informed investors hold in the first period will be cleared by second period informed investors. This, in turn, amplifies both the inventory risk and the adverse selection risk to which first period investors are exposed when liquidating, making p_2 less predictable and leading them to lower their response to private information. Along the low illiquidity equilibrium, more persistent liquidity trading means that more informed investors are escalating their responsiveness to private information, making p_2 more dependent on p_1. As a consequence, individually each trader scales down his reliance on private information. Note that as $\beta \to 0$, along the high illiquidity equilibrium a_1 converges (its expression is given by (22)), while along the low illiquidity equilibrium a_1 diverges. Intuitively, along the low illiquidity equilibrium, the smaller is β, the lower is the number of first period informed investors who cannot count on the reversion of liquidity traders to unwind their positions. As a consequence, the more aggressively each informed investor needs to respond to private information in the first period for second period investors to lower their exposure to the risky asset.

We conclude this section by analyzing the expected losses liquidity traders incur along the two equilibria. If $\beta = 0$ all first period liquidity traders unwind their position, so that in the second period there is more liquidity trading. At the other extreme, if $\beta = 1$, the total amount of second period liquidity trading is much smaller, as first and second period investors share the additional shock u_2. In general for any $\beta \in [0, 1]$, the expected profit that the first period liquidity traders obtain is thus given by

$$\begin{aligned} \Pi_{\theta_1}(a_1, \beta) &\equiv E[\beta\theta_1(v - p_1) + (1-\beta)\theta_1(p_2 - p_1)] \\ &= -\left(\beta\lambda_1 + (1-\beta)\lambda_2\Delta a_2 \left(\frac{\tau_{i1} - \tau_v}{a_1\tau_{i1}}\right)\right)\tau_u^{-1} < 0. \end{aligned} \tag{23}$$

It is easy to see that along the low illiquidity equilibrium $\lim_{\beta\to 0} \Pi_{\theta_1} = 0$, since in this case the first period signal responsiveness diverges at $\beta = 0$ (see the discussion following Corollary 6). On the contrary, along the equilibrium with high illiquidity

$$\lim_{\beta\to 0} \Pi_{\theta_1} = -\frac{(1 + \gamma a_2\tau_u)^2(\tau_{i1} - \tau_v)}{\gamma^2 a_2\tau_u\tau_{i1}\tau_{i2}} < 0.$$

In general, plotting (23) along the two equilibria that arise with $\beta > 0$ one obtains Figure 2. Depending on parameters values the two plots intersect or not (we need τ_v high for them to intersect), but the bottomline is that for β small, liquidity traders' expected losses are always smaller along the low illiquidity equilibrium (in the second period the conclusion is immediate given the properties of the equilibrium).

4.1 An additional trading round

If we add an extra trading round to the market analyzed in Section 4, the coordination problem across different cohorts exacerbates, and the cardinality of the equilibrium set increases. Intuitively, in the second period rational investors anticipate either a liquid or an illiquid third period market. Accordingly, two different levels of second period

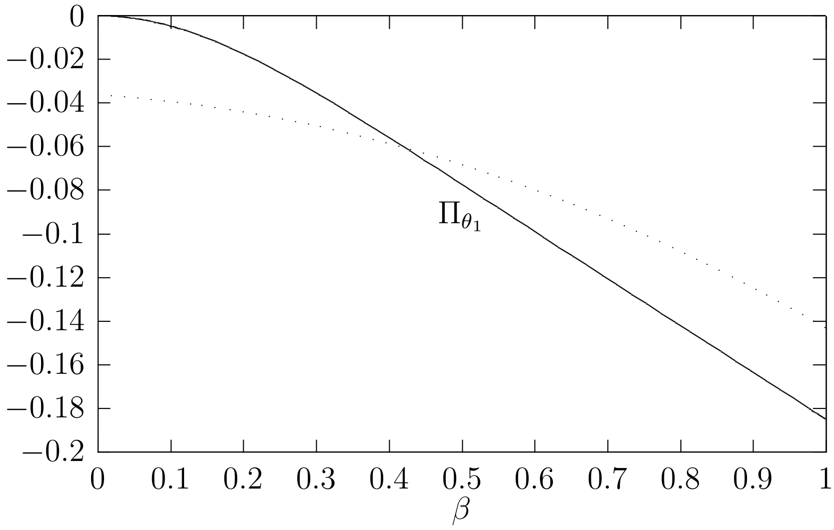

Figure 2: Expected profit along the high illiquidity equilibrium (dotted line) and along the low illiquidity equilibrium (continuous line) as a function of β. Other parameter values are $\tau_v = 10$, $\tau_u = 1$, $\tau_\epsilon = 1$, $\gamma = 1$.

equilibrium signal responsiveness arise. One period before, the same problem is now faced by first period investors *for each* equilibrium on which second period investors coordinate. This gives rise to four equilibria: first period investors anticipate low second and third period illiquidity; alternatively, they anticipate low illiquidity in the second, followed by high illiquidity in the third period, or high illiquidity in the second followed by low illiquidity in the third period; finally, they may anticipate that illiquidity will stay high in both the second and third periods. Correspondingly, four different levels of first period signal responsiveness can arise. Figure 3 illustrates our findings.

5 Illiquidity and reliance on public information

In this section we investigate the implications that heterogeneous trading frequencies have for price reliance on public information. We start by obtaining a general expression for the equilibrium price which shows that in general asset prices are driven by investors'

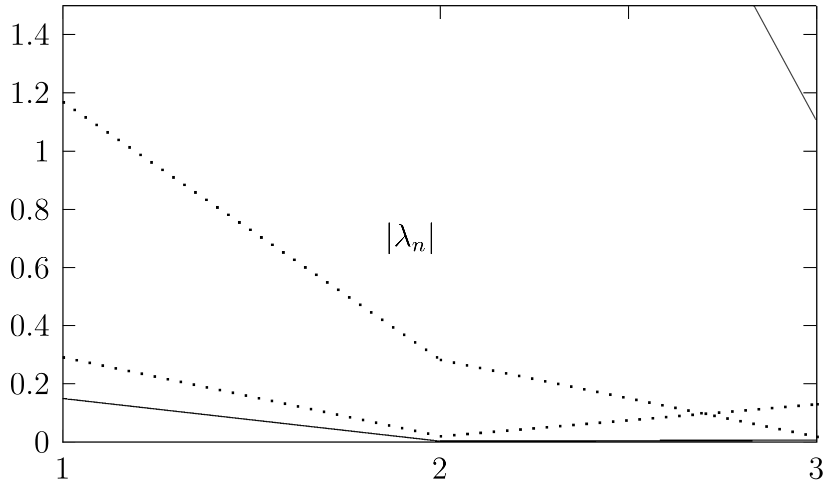

Figure 3: The figure displays $|\lambda_n|$ for $n = 1, 2, 3$, along the four equilibria. The thick (thin) line shows the time path of $|\lambda_n|$ when investors anticipate that the market will be very liquid (illiquid) at both dates 2 and 3 (along the high illiquidity equilibrium $\lambda_1 = 26.9$ and $\lambda_2 = 3.4$). The dotted lines reflect the situation in which investors' expectations about future illiquidity alternate. Parameter values are as follows: $\tau_v = \tau_\epsilon = \tau_u = 1$, $\gamma = .5$, and $\beta = .8$.

HOEs about the two factors that influence the aggregate demand: fundamentals and liquidity trades.

Starting from the third period, and imposing market clearing yields

$$\int_0^1 X_3(s_{i3}, p^3)di + \theta_3 = 0. \tag{24}$$

At any linear equilibrium, the price will be a normally distributed random variable, which, owing to the fact that investors' utility displays CARA implies that

$$X_3(s_{i3}, p^3) = \gamma \frac{E_{i3}[v] - p_3}{\text{Var}_{i3}[v]}.$$

Replacing the above in (24) and solving for the equilibrium price we obtain

$$p_3 = \bar{E}_3[v] + \Lambda_3\theta_3,$$

where $\Lambda_3 = \text{Var}_{i3}[v]/\gamma$. Similarly, in the second period, imposing market clearing yields:

$$\int_0^1 X_2(s_{i2}, p^2)di + \theta_2 = 0,$$

and solving for the equilibrium price we obtain

$$p_2 = \bar{E}_2[p_3] + \frac{\text{Var}_{i2}[p_3]}{\gamma}\theta_2. \tag{25}$$

Substituting the above obtained expression for p_3 in (25) yields

$$\begin{aligned} p_2 &= \bar{E}_2\left[\bar{E}_3[v] + \frac{\text{Var}_{i3}[v]}{\gamma}\theta_3\right] + \frac{\text{Var}_{i2}[p_3]}{\gamma}\theta_2 \\ &= \bar{E}_2\left[\bar{E}_3[v]\right] + \frac{\text{Var}_{i3}[v]}{\gamma}\beta\bar{E}_2\left[\theta_2\right] + \frac{\text{Var}_{i2}[p_3]}{\gamma}\theta_2. \end{aligned} \tag{26}$$

According to (26), there are three terms that form the second period price: investors' second order average expectations over the liquidation value ($\bar{E}_2[\bar{E}_3[v]]$), the risk-adjusted impact of the second period stock of liquidity trades (θ_2), *and* investors' average expectations over second period liquidity trades ($\bar{E}_2[\theta_2]$). As liquidity trades are persistent, rational investors anticipate unwinding a fraction β of their inventory (θ_2) to third period investors, thereby affecting p_3. Due to heterogeneous information, however, θ_2 cannot be perfectly assessed. Thus, p_2 reflects the second period market consensus over the size of liquidity traders' demand.

In the first period, a similar argument yields

$$p_1 = \bar{E}_1\left[\bar{E}_2\left[\bar{E}_3\left[v\right]\right]\right] + \frac{\mathrm{Var}_{i3}[v]}{\gamma}\beta\bar{E}_1\left[\bar{E}_2[\theta_2]\right] + \frac{\mathrm{Var}_{i2}[p_3]}{\gamma}\beta\bar{E}_1[\theta_1] + \frac{\mathrm{Var}_{i1}[p_2]}{\gamma}\theta_1, \quad (27)$$

and generalizing (27) to an arbitrary number of periods N we obtain

$$\begin{aligned} p_n = {} & \bar{E}_n\left[\bar{E}_{n+1}\left[\bar{E}_{n+2}\left[\cdots\bar{E}_N[v]\cdots\right]\right]\right] \quad (28) \\ & + \frac{\beta}{\gamma}\mathrm{Var}_{iN}[p_{N+1}]\bar{E}_n\left[\bar{E}_{n+1}\left[\bar{E}_{n+2}\left[\cdots\bar{E}_{N-1}[\theta_{N-1}]\cdots\right]\right]\right] \\ & + \frac{\beta}{\gamma}\mathrm{Var}_{iN-1}[p_N]\bar{E}_n\left[\bar{E}_{n+1}\left[\bar{E}_{n+2}\left[\cdots\bar{E}_{N-2}[\theta_{N-2}]\cdots\right]\right]\right] \\ & + \cdots \\ & + \frac{\mathrm{Var}_{in}[p_{n+1}]}{\gamma}\theta_n. \end{aligned}$$

The above expression shows that in period n the equilibrium price reflects investors' HOEs over the liquidation value *and* over the liquidity trades in periods $n, n+1, \ldots, N-1$. The former factor reflects the findings of Morris and Shin (2002), and Allen et al. (2006) who prove that when investors have heterogeneous information, the law of iterated expectations fails to hold:

$$\bar{E}_n\left[\bar{E}_{n+1}\left[\bar{E}_{n+2}\left[\cdots\bar{E}_N[v]\cdots\right]\right]\right] \neq \bar{E}_n[v].$$

When prices are only driven by HOEs over the final payoff, they are systematically farther away from fundamentals compared with consensus or, equivalently, they heavily rely on public information (compared to optimal statistical weights). In our context, the presence of asynchronous liquidity needs implies that an additional factor adds to the weight contributed by HOEs. Computing the expectations (26) and (27) we obtain

$$\bar{E}_2\left[\bar{E}_3[v]\right] = \bar{\alpha}_{E_2} v + (1 - \bar{\alpha}_{E_2})\, E_2[v], \;\; \bar{E}_1\left[\bar{E}_2\left[\bar{E}_3\left[v\right]\right]\right] = \bar{\alpha}_{E_1} v + (1 - \bar{\alpha}_{E_1})\, E_1[v],$$

where

$$\bar{\alpha}_{E_1} = \alpha_{E_1}\left(1 - \frac{\tau_1}{\tau_2}(1 - \bar{\alpha}_{E_2})\right), \;\; \bar{\alpha}_{E_2} = \alpha_{E_2}\left(1 - \frac{\tau_2}{\tau_3}(1 - \alpha_{E_3})\right), \tag{29}$$

denote the weights that HOEs about the final payoff assign to v in the first and second period price, and $\tau_n = 1/\mathrm{Var}[v|p^n]$. Similarly,

$$\bar{E}_n[\theta_n] = a_n(1 - \alpha_{E_n})(v - E_n[v]) + \theta_n$$
$$\bar{E}_1[\bar{E}_2[\theta_2]] = (a_2(\alpha_{E_1} - \bar{\alpha}_{E_1}) + \beta a_1(1 - \alpha_{E_1}))(v - E_1[v]) + \beta\theta_1.$$

According to Proposition 2, at any linear equilibrium, the price can be expressed as follows:

$$p_n = \alpha_{P_n}\left(v + \frac{\theta}{a_n}\right) + (1 - \alpha_{P_n})\, E_n[v].$$

Hence, we obtain:

Lemma 1. *When $\beta > 0$, the weights the price assigns to the fundamentals in the first and second period are given by*

$$\alpha_{P_1} = \bar{\alpha}_{E_1} + \beta\left((1 - \alpha_{E_1})\Lambda_2 a_1 + (\alpha_{E_1} - \bar{\alpha}_{E_1})\,\Lambda_3 a_2\right) \tag{30}$$

$$\alpha_{P_2} = \bar{\alpha}_{E_2} + \beta(1 - \alpha_{E_2})\Lambda_3 a_2. \tag{31}$$

Similarly as in Allen et al. (2006), we say that at time n the price is systematically farther away from investors' consensus opinion if the following condition holds true:

$$|E\left[p_n - v|v\right]| > \left|E\left[\bar{E}_n[v] - v|v\right]\right|. \tag{32}$$

The above condition holds if, for any liquidation value, averaging out the impact of noise trades, the discrepancy between the price and the fundamentals is always larger than that between investors' average opinion and the fundamentals. Using the expression obtained in Proposition 2, the following result offers two alternative characterizations of condition (32).

Lemma 2. *At any linear equilibrium of the* 3*-period market the following three conditions are equivalent:*

$$|E[p_n - v|v]| > \left|E\left[\bar{E}_n[v] - v|v\right]\right| \tag{33}$$

$$\alpha_{P_n} < \alpha_{E_n} \tag{34}$$

$$\mathrm{Cov}[p_n, v] < \mathrm{Cov}\left[\bar{E}_n[v], v\right]. \tag{35}$$

Thus, as intuition suggests, the equilibrium price is systematically farther away from fundamentals compared to consensus, whenever the price overweights public information (compared to optimal statistical weights); equivalently, whenever the price scores worse than investors' average opinion in predicting the fundamentals.

When $\beta = 0$, it is easy to see that $\alpha_{P_1} = \bar{\alpha}_{E_1} < \alpha_{E_1}$, so that over-reliance on public information occurs (compared to the optimal statistical weight). However, as we argued is Proposition 3, when $\beta > 0$, $\alpha_{P_1} < \alpha_{E_1}$ if and only if $a_1 = a_1^*$. Thus, we can immediately conclude

Proposition 4. *When $N = 2$ and $\beta > 0$, along the equilibrium with high illiquidity the first period price over-relies on public information (compared to the optimal statistical* *weight). Conversely, along the equilibrium with low illiquidity, the price under-relies on public information (compared to the optimal statistical weight).*

Thus, the existence of different trading frequencies (due to persistence in liquidity trades) generate an effect that can *offset* the gravitational pull towards over-reliance on public information due to HOEs over fundamentals. This effect works precisely by forcing investors to internalize the negative impact that their position unwinding has on future periods' illiquidity. Thus, short-term investment horizons per-se do not warrant over-reliance on public information. It is rather a matter of how heterogeneous investment horizons are across agents' types. The need to forecast a price that reflects investors' valuations which are also based on a public signal (the equilibrium price), leads to over-reliance on public information. On the other hand, the need to minimize the illiquidity cost borne when unwinding their positions, pushes investors to over-rely on their private signals. Along the equilibrium with high illiquidity, the former effect prevails and the price is driven by HOEs over the final payoff. Conversely, along the equilibrium with low illiquidity, HOEs over the payoff are subdued and prices track fundamentals more efficiently. Our analysis thus portrays a more complex picture of Keynes' Beauty Contest asset pricing allegory.

Numerical simulations confirm the relationship between the high illiquidity equilibrium and over-reliance on public information also in the 3-period extension presented in Section 4.1. We illustrate this finding in Figure 4, where we add the plots of $\alpha_{P_n} - \alpha_{E_n}$ along the 4 equilibria that arise in this case to the plots of illiquidity presented in Figure 3.

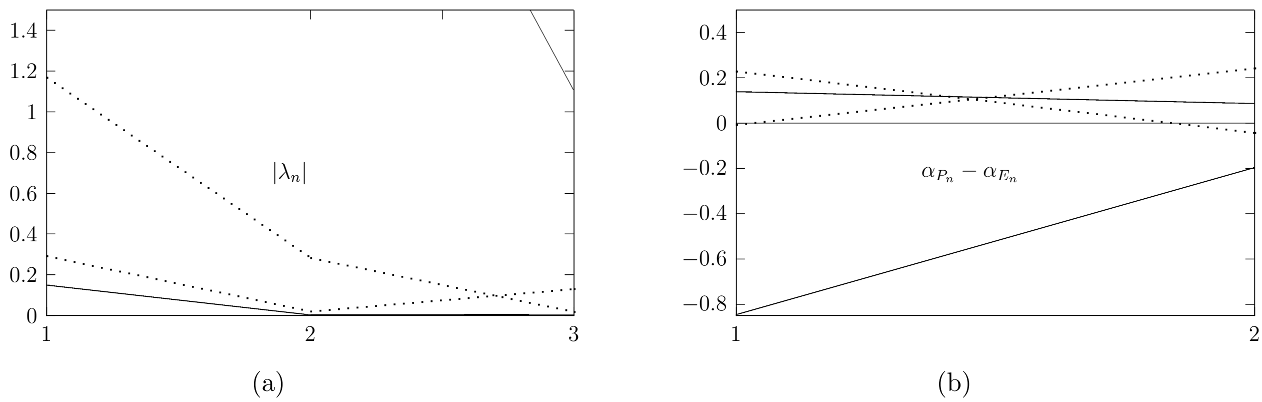

Figure 4: In panel (a) we plot $|\lambda_n|$ for $n = 1, 2, 3$, and in panel (b) $\alpha_{P_n} - \alpha_{E_n}$ for $n = 1, 2$, along the four equilibria. The solid lines in panel (b) correspond to equilibria in which the market relies heavily (moderately) on public information in both periods. Correspondingly, the solid lines in panel (a) show that when the market relies heavily (moderately) on public information illiquidity is high (low) in both periods (along the high illiquidity equilibrium $\lambda_1 = 26.9$ and $\lambda_2 = 3.4$). The dotted lines in both panels correspond to equilibria in which the market relies heavily (moderately) on public information in the first (second) period and moderately (heavily) on public information in the second (first) period implying an illiquidity pattern of high (low) illiquidity in the first period, low (high) illiquidity in the second period and finally high (low) illiquidity in the third period. Other parameter values are as follows: $\tau_v = \tau_\epsilon = \tau_u = 1$, $\gamma = .5$, and $\beta = .8$.

6 Asset pricing implications

In this section we discuss the implications of our results for the predictability of returns, the evolution of the volume of informational trading, and for the effect of illiquidity on expected returns.

6.1 Momentum and reversal

We start our analysis by computing the autocovariance of returns at different trading horizons. A first implication of our model is that return autocorrelation depends on trading horizons, *and* on the equilibrium on which investors coordinate:

Corollary 7. *When $N = 2$:*

1. *For all* $\beta \in [0,1]$, $\mathrm{Cov}[p_2 - p_1, p_1 - \bar{v}] < 0$.

2. *For* $\beta \in (0,1]$, $\mathrm{Cov}[v - p_2, p_1 - \bar{v}] < 0$. *For* $\beta = 0$, $\mathrm{Cov}[v - p_2, p_1 - \bar{v}] = 0$.

3. *For* $\beta \in (0,1]$, *along the equilibrium with low illiquidity* $\mathrm{Cov}[v - p_2, p_2 - p_1] > 0$. *Along the equilibrium with high illiquidity, for* $\tau_v < \hat{\tau}_v$, *there exists a value* $\hat{\beta}$ *such that for all* $\beta > \hat{\beta}$, $\mathrm{Cov}[v - p_2, p_2 - p_1] > 0$ *(the expression for* $\hat{\tau}_v$ *is given in the appendix, see equation* (A.52)*). If* $\beta = 0$, $\mathrm{Cov}[v - p_2, p_2 - p_1] < 0$.

According to the above result, along the equilibrium with low illiquidity, momentum occurs at short horizons (close to the end of the trading horizon), whereas at a longer horizon, returns display reversal.[14] This is in line with the empirical findings on return anomalies that document the existence of positive return autocorrelation at short horizons (ranging from six to twelve months, see Jegadeesh and Titman (1993)), and negative autocorrelation at long horizons (from three to five years, see De Bondt and Thaler (1985)).

It is interesting to relate this result with Daniel, Hirshleifer, and Subrahmanyam (1998) who assume that *overconfident* investors underestimate the dispersion of the error term affecting their signals and "overreact" to private information. This, in turn, generates long term reversal and, in the presence of confirming public information which due to *biased self attribution* boosts investors' confidence, also lead to short term positive return autocorrelation. This pattern of overreaction, continuation, and correction is likely to affect stocks which are more difficult to value (e.g., growth stocks). In such a context, momentum is thus a symptom of mispricing and hence related to prices wandering away from fundamentals (conversely, reversal is identified with price corrections). In our model, along the equilibrium with low illiquidity, investors rationally react more

[14] The fact that $\mathrm{Cov}[p_2 - p_1, p_1 - \bar{v}] < 0$ is due to the "initial effect," $p_0 = \bar{v}$. Numerical simulations show that in a model with three periods, in the equilibrium with low illiquidity, both $\mathrm{Cov}[v - p_3, p_3 - p_2]$ and $\mathrm{Cov}[p_3 - p_2, p_2 - p_1]$ are positive.

strongly to their private signals compared to the static benchmark, in contrast to the overreaction effect of the behavioral literature.[15] However, this heavy reaction to private information leads to stronger information impounding and to prices that track better the fundamentals (see Proposition 3). Momentum at short horizons in this case is therefore associated with a rapid convergence of the price to the full information value. To illustrate this fact, in Figure 5 we plot the mean price paths along the equilibrium with high illiquidity (thick line), with low illiquidity (thin line), and along the "static" equilibrium, that is the one that would obtain if investors reacted to information as if they were in a static market (dotted line). From the plot it is apparent that in the equilibrium with low illiquidity the price displays a faster adjustment to the full information value than in the equilibrium with high illiquidity (and the static equilibrium). This shows that the occurrence of momentum is not at odds with price efficiency.

[15]Indeed, the static solution calls for $a_1 = \gamma\tau_\epsilon$ (see, e.g, Admati (1985), or Vives (2008)), and it is easy to verify that $0 < a_1^* < \gamma\tau_\epsilon < a_1^{**}$. In Daniel, Hirshleifer, and Subrahmanyam (1998) overconfident investors overweight private information in relation to the prior.

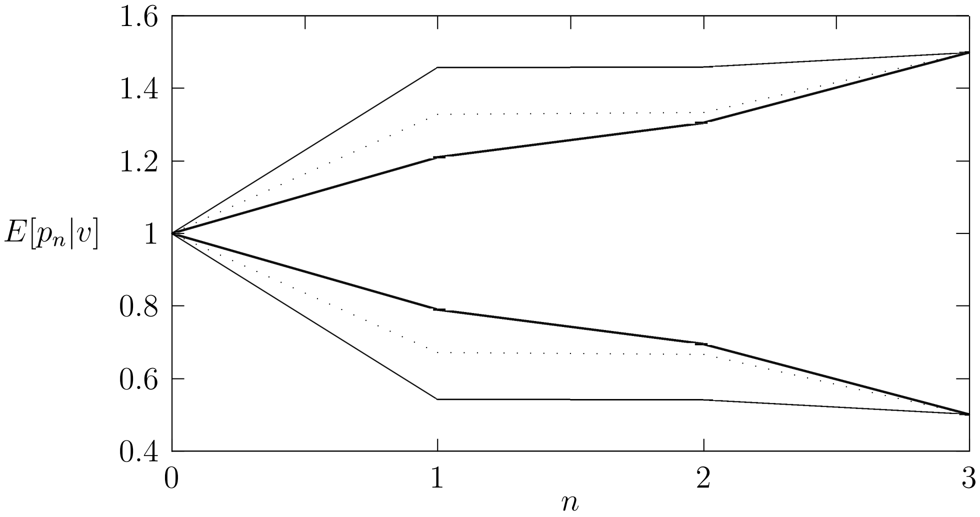

Figure 5: Mean price paths along the equilibrium with high illiquidity (thick line), low illiquidity (thin line), and assuming that first period investors react to private information as if they were in a static market (i.e., setting $a_1 = \gamma\tau_\epsilon$). Parameters' values are as follows: $\tau_v = \tau_\epsilon = \tau_u = \gamma = 1$, $\bar{v} = 1$, $\beta = .9$ and $v \in \{1.5, .5\}$.

As stated in the corollary, momentum can also occur along the equilibrium with high illiquidity, provided that investors are sufficiently uncertain about the liquidation value prior to trading (that is, τ_v is low) and that liquidity trading is sufficiently persistent (β high). In that equilibrium, investors respond less to private information, information impounding is staggered, and prices adjust more slowly to the full information value (see Figure 5). However, as observed in Section 2, if sufficiently persistent, liquidity trading exerts a continuous price pressure which can eventually outweigh the former effect. Therefore, along this equilibrium momentum arises with slow convergence to the full information value, implying that the occurrence of a positive autocorrelation at short horizons *per se* does not allow to unconditionally identify the informational properties of prices.

Comparing the parameter space that yields momentum in the absence of private information (Section 2) with the one obtained in Corollary 7 along the equilibrium with high illiquidity, allows to identify the effect of heterogeneous information in generating return continuation. In Figure 6 we plot the parameter space that yields momentum in the absence of private information (light and dark gray) and the one yielding momentum with private information along the equilibrium with high illiquidity (dark gray) for $\tau_v \in [0.001, 10]$, $\beta \in [0, 1]$, $\gamma = \tau_\epsilon = 1$, and $\tau_u = 0.5$. From the figure it is apparent that private information along the high illiquidity equilibrium moderates the impact of persistent liquidity trading.[16] Therefore, the gradual impounding of private information in this case is not conducive to momentum (as opposed to Hong and Stein (1999)). The figure stresses the fact that for momentum to arise, τ_v needs to be small, that is the unconditional uncertainty about the liquidation value faced by investors needs to be sufficiently large. As argued above, this prediction is consistent with the findings of the behavioral finance literature, which associates momentum to overreaction (Daniel, Hirshleifer, and Subrahmanyam (1998)). Note however, that in the high illiquidity equilibrium momentum occurs because of persistent liquidity trading and in the absence of a behavioral bias.

Finally, at long horizons, the effect of private information on the correlation of returns washes out and, as in the market with homogeneous information (see Corollary 1), the only driver of the autocovariance is the persistence in liquidity trading, which generates reversal.

6.2 Expected volume and return predictability

In this section we investigate the implications of our results for the expected volume of informational trading and the predictability of returns along the two equilibria that arise with $N = 2$. We show that the expected volume of informational trading is high (low) along the low (high) illiquidity equilibrium. This implies that a high volume of infor-

[16] Numerical simulations show that as τ_u increases, the yellow region disappears, while the blue region remains even for relatively high values of the precision of liquidity trading (e.g., $\tau_u = 4$).

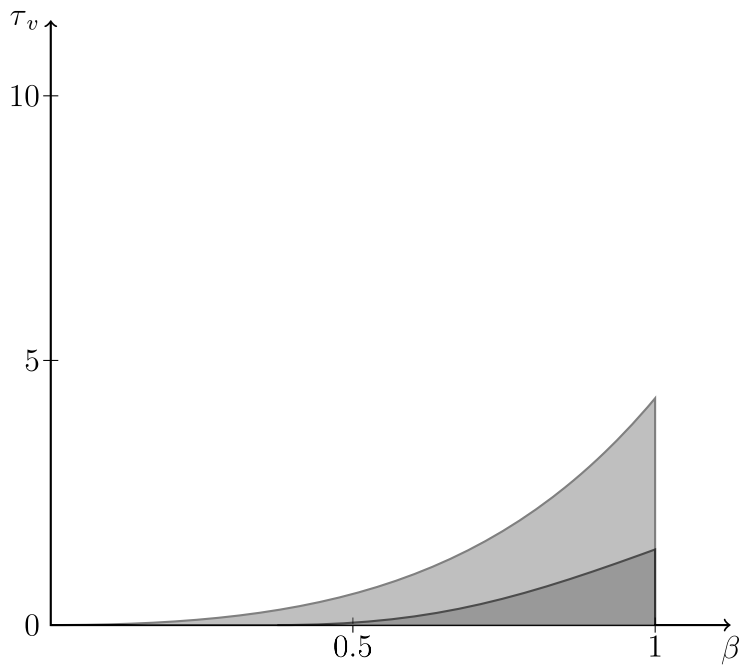

Figure 6: The light and dark gray regions capture the space of values (β, τ_v) for which momentum occurs in the model with homogeneous information. The dark gray region captures the space of values (β, τ_v) for which momentum occurs in the model with heterogeneous information along the equilibrium with high illiquidity. Other parameter values: $\gamma = \tau_\epsilon = 1$, and $\tau_u = .5$.

mational trading predicts momentum, in line with the evidence presented by Llorente, Michaely, Saar, and Wang (2002). However, as we have argued in the previous section, also along the equilibrium with high illiquidity momentum can occur, provided liquidity trading displays sufficiently strong persistence (and the ex-ante uncertainty about the liquidation value is sufficiently high). This implies that a low volume of informational trading can also predict continuation. In this case, though, momentum is a signal of slow price convergence to the liquidation value. In sum, momentum is compatible with both a high and a low volume of informational trading, but the implications that return continuation has for price informativeness are markedly different in the two situations.

We start by defining the volume of informational trading as the expected traded volume in the market with heterogeneous information net of the expected volume that obtains in the market with no private information analyzed in Section 2. This yields:[17]

$$
\begin{aligned}
V_2 &\equiv \int_0^1 E\left[\left|X_2(s_{i2}, z^2) - X_1(s_{i1}, z_1)\right|\right] di - \int_0^1 E\left[\left|X_2(\theta_2) - X_1(\theta_1)\right|\right] di \\
&= \int_0^1 \sqrt{\frac{2}{\pi}\mathrm{Var}\left[X_2(s_{i2}, z^2) - X_1(s_{i1}, z_1)\right]} di - \int_0^1 \sqrt{\frac{2}{\pi}\mathrm{Var}\left[X_2(\theta_2) - X_1(\theta_1)\right]} di \\
&= \sqrt{\frac{2}{\pi}}\left(\sqrt{(a_1^2 + a_2^2)\tau_\epsilon^{-1} + (1 + (\beta - 1)^2)\tau_u^{-1}} - \sqrt{(1 + (\beta - 1)^2)\tau_u^{-1}}\right). \qquad (36)
\end{aligned}
$$

Corollary 8. *When $N = 2$, for all $\beta \in (0, 1]$ the expected volume of informational trading is higher along the low illiquidity equilibrium. When $\beta = 0$ only the equilibrium with a low volume of informational trading survives.*

Proof. Rearranging the expressions for investors' strategies obtained in Corollary 5 yields $x_{in} = a_n \epsilon_{in} - \theta_n$, for $n = 1, 2$. Owing to the fact that for a normally distributed random variable Y we have

$$
E[|Y|] = \sqrt{\frac{2}{\pi}\mathrm{Var}[Y]},
$$

which implies (36), an increasing function of a_1. Recall that while $a_2 = \gamma\tau_\epsilon$, in the first period the response to private information is higher along the equilibrium with low illiquidity: $a_1^{**} > a_1^*$, and the result follows. Finally, from Proposition 3 when $\beta = 0$,

$$
a_1 = \frac{\gamma a_2^2 \tau_u}{1 + \gamma a_2 \tau_u} < a_1^{**}.
$$

□

The intuition for the above result is straightforward: as along the equilibrium with low illiquidity investors step up the response to their signals, the position change due to private information is higher along such equilibrium.

Taking together Corollary 7 and 8 implies that a high volume of informational trading

[17] This is consistent with He and Wang (1995).

in the second period predicts return continuation, no matter what the persistence in liquidity trading is. A low volume of informational trading, on the other hand, can also be associated with momentum, provided liquidity trading is sufficiently persistent.

6.3 Illiquidity and expected returns

Several contributions find that illiquidity influences asset prices and commands a premium for asset expected returns (see Amihud and Mendelson (1986), Easley and O'Hara (2004), Amihud, Mendelson, and Pedersen (2005), and Vayanos and Wang (2009)). In this section we study the implications of our analysis for the pricing of illiquidity. As stressed in the previous sections, in the presence of heterogeneous trading frequencies ($\beta > 0$), our model delivers multiple equilibria that can be ranked in terms of illiquidity. This implies that if investors ex-ante expect the stock of liquidity trading to be positive, a "liquidity premium" will appear in the asset pricing equation whose magnitude is directly related to the equilibrium on which the market coordinates. To see this, we use (10) and compute the n-th short-term unconditional expected return. Due to the law of iterated expectations, this yields:

$$\begin{aligned} E[p_n - p_{n-1}] &= E\left[E_n[v] + \Lambda_n E_n[\theta_n] - E_{n-1}[v] - \Lambda_{n-1} E_{n-1}[\theta_{n-1}]\right] \\ &= \Lambda_n E[\theta_n] - \Lambda_{n-1} E[\theta_{n-1}]. \end{aligned} \tag{37}$$

Assuming $u_n \sim N(\bar{u}, \tau_u^{-1})$, with $\bar{u} > 0$, implies that (37) is non null, and shows that the unconditional risk premium is driven solely by the inventory risk component of market illiquidity Λ_n. We formalize our results in the following

Corollary 9. *When $N = 2$, and $u_n \sim N(\bar{u}, \tau_u^{-1})$,*

$$E[p_1 - \bar{v}] = \Lambda_1 \bar{u} \tag{38}$$

$$E[p_2 - p_1] = (\Lambda_2(1+\beta) - \Lambda_1)\, \bar{u} \tag{39}$$

$$E[v - p_2] = -\Lambda_2(1+\beta)\bar{u}. \tag{40}$$

Denoting by Λ_n^H and Λ_n^L, respectively the inventory component of market illiquidity associated with the high and low illiquidity equilibrium at time n it is possible to show the following

Corollary 10. *When $N = 2$, $\Lambda_n^L < \Lambda_n^H$, for $n = 1, 2$.*

Thus, as investors anticipate worse market conditions along the high illiquidity equilibrium, the expected compensation required to absorb the demand shock from liquidity traders in the first and second period is higher, implying a higher risk premium for first and third period returns. Numerical simulations show that a similar result also holds in the second period. In Figure 7 we assume $\bar{u} = 1$, and plot the risk premia associated with short term returns along the high and low illiquidity equilibrium as a function of the heterogeneity in trading frequencies (β). Panel (b) of the figure, displays the comparison for the illiquidity premia associated with the second period returns, and shows that the premium is higher along the equilibrium with high illiquidity. Thus, for $\beta > 0$, the risk premium is higher along the high illiquidity equilibrium at all trading dates.

A further implication of the figure is that as β increases from 0 to 1, the risk premia at all trading dates increase, along each equilibrium. Intuitively, as β becomes larger, a higher fraction of liquidity traders' positions at any date remains in the market, augmenting the inventory risk born by rational investors. As a consequence, the expected premium required to absorb liquidity trades increases.

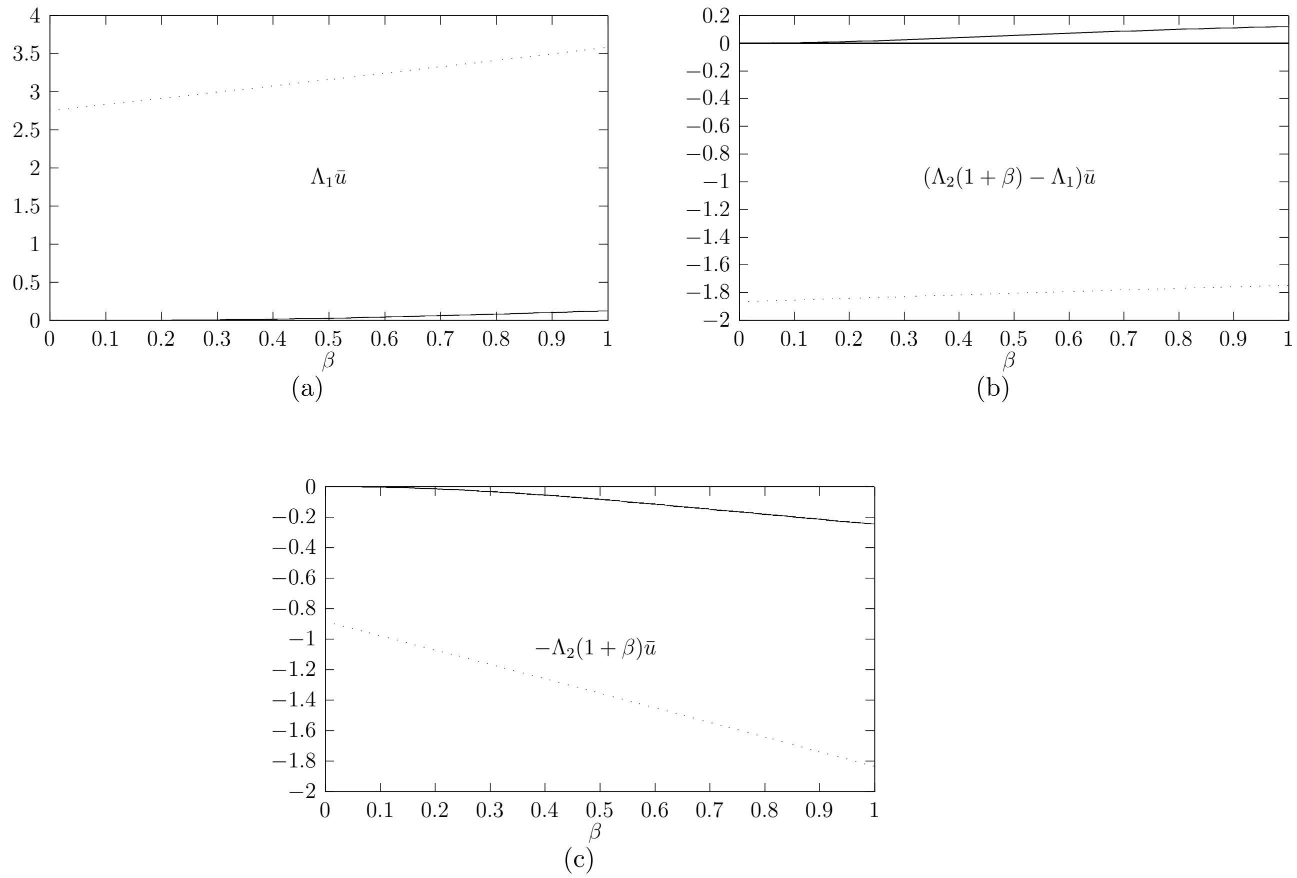

Figure 7: In panel (a), (b), and (c) we plot the risk premium associated with first, second, and third period returns along the high and low illiquidity equilibrium (respectively, dotted and continuous line) for $\beta \in [0, 1]$. Parameters' values are as follows: $\tau_v = \tau_u = \tau_\epsilon = \bar{u} = 1$, and $\gamma = 1/2$.

It is interesting to relate our results with Easley and O'Hara (2004) who show that private information induces a risk that is priced in equilibrium. In their setup, stocks for which the amount of private information is higher (compared to public signals) command a premium that serves to compensate uninformed investors for the losses they expect to make vis-à-vis informed investors because of adverse selection. A similar effect is also at work in our model. Indeed, although the illiquidity premium is only related to the inventory risk component of illiquidity Λ_n (and thus the adverse selection component of illiquidity does not appear in (37)), as we have argued in Section 4, along the high illiquidity equilibrium investors in period 1 anticipate that their next period peers will increase their exposure to the risky asset, thereby inducing an adverse selection problem at the liquidation date. This will make the interim liquidation price more volatile, increasing price risk and lowering the first period response to private information. Thus, in this equilibrium, the higher expected returns occur together with a more severe adverse selection problem.

Notice finally, that according to panel (b) in Figure 7 the unconditional expectation of the second period return has an opposite sign along the two equilibria that arise when $\beta > 0$. In the high (low) illiquidity equilibrium $E[p_2] < E[p_1]$ ($E[p_2] > E[p_1]$). This finding is in line with our intuition for the two equilibria: in the high illiquidity equilibrium, the aggregate demand is driven mainly by liquidity traders, which implies that when investors expect a positive stock of liquidity trading, they anticipate reversal. Conversely, in the low illiquidity equilibrium, as the aggregate demand is mainly driven by informed trading, in the same condition, continuation is expected.

To summarize, liquidity trading persistence affects the expected returns in *two* distinct ways. On the one hand, for a *given positive level* of liquidity demand persistence, multiple equilibria arise which can be ranked in terms of illiquidity. Along the equilibrium with high illiquidity, the premium required to absorb the demand shock from liquidity traders is higher, implying higher expected returns. On the other hand, an *increase in persistence* augments the risk born by rational investors. This increases the inventory risk component of market illiquidity and leads to higher expected returns along *each equilibrium.*

Table 2 collects our results stressing the interplay between illiquidity, price informativeness, volume, the impact of HOEs on asset prices, the correlation of returns at different horizons, and expected returns.

	$\beta = 0$	$\beta \in (0, 1]$	
		High Illiq. Eq.	Low Illiq. Eq.
Reliance on public information	$\alpha_{P_1} < \alpha_{E_1}$	$\alpha_{P_1} < \alpha_{E_1}$	$\alpha_{P_1} > \alpha_{E_1}$
Illiquidity	High	High	Low
Expected volume of informational trading	Low	Low	High
Price informativeness	Low	Low	High
Return correlation at long horizons	0	$-$	$-$
Return correlation at short horizons	$-$	$\pm$	$+$
Expected returns	High	High	Low

Table 2: A summary of our results. When $\beta = 0$ liquidity traders' demand is transient while when $\beta > 0$ it is persistent.

7 Conclusions

When a market is populated by short term investors, the persistence of liquidity traders' demand implies that only a fraction of the latters' orders is bound to revert when informed investors close out their positions. Hence, each cohort of investors unwinds part of its holdings against the aggregate demand coming from the next period cohort, thereby shifting part of the risk it incorporates and potentially facing an adverse selection problem at the liquidation date. This makes portfolio decisions more sensitive to the illiquidity of the market in which investors plan to unwind their positions. With heterogeneous information, a more illiquid market in the future, lowers price dependence on fundamentals, reducing investors' reliance on their private information. Conversely, when investors anticipate a less illiquid future market they speculate more aggressively on their private signals. Thus, with persistent liquidity trading and heterogeneous infomation, short term horizons deliver multiple equilibria which can be ranked in terms of illiquidity. A first implication of our model is thus that with differential information, heterogenous trading frequencies make illiquidity dependent on a coordination problem across different generations of investors, and thereby can endogenously create illiquidity risk.

We show that along the equilibrium in which investors anticipate higher future illiquidity, prices are driven by HOEs about fundamentals, and therefore over-rely on public information (compared to the optimal statistical weight). In this equilibrium investors scale down their response to private signals yielding an increase in price risk, which leads to an increase in the premium required to absorb the demand shock from liquidity traders, and implies higher expected returns. When instead investors anticipate a less illiquid market in the future, they step up their response to private signals, reducing price risk and expected returns.

Our paper provides an alternative interpretation for empirically documented regularities on the patterns of return autocorrelation. As we have argued, at long horizons, returns display reversal. However, return correlation at short horizons depends on the equilibrium that prevails in the market. In the equilibrium with low illiquidity, investors step up their response to private information and momentum arises. Conversely, in the high illiquidity equilibrium investors scale down their response to private signals and, when liquidity trading is not very persistent, returns tend to revert. While this offers an explanation for returns' predictability which departs from behavioral assumptions, our analysis also makes the empirical prediction that both a high or a low volume of informational trading can predict momentum. In the former case, this is a signal that prices rely poorly on public information and accurately reflect fundamentals starting from the earlier stages of the trading process. In this case momentum at short horizons proxies for a rapid price convergence to the full information value. In the latter case, instead, prices heavily rely on public information and offer a poor signal of fundamentals. In this case, therefore, momentum proxies for a continuing, liquidity-driven, price pressure.

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A Appendix

<u>Proof of Proposition 1</u>

Assume that in a linear equilibrium $x_n = -\phi_n(p_n)$, with $\phi_n(\cdot)$ a linear function of p_n. This implies that the market clearing equation at time n reads as follows:

$$x_n + \theta_n = 0 \Leftrightarrow -\phi_n(p_n) + \theta_n = 0.$$

Hence, at equilibrium p_n is observationally equivalent to θ_n, i.e. at time n rational investors *know* the realisation of the noise stock θ_n. To solve for the equilibrium we proceed by backward induction and start from the third trading round, where due to CARA and normality, we have

$$\begin{aligned} X_3(p_3) &= \gamma \frac{E_3[v] - p_3}{\text{Var}_3[v]} \\ &= -\Lambda_3^{-1}(p_3 - \bar{v}), \end{aligned} \tag{A.1}$$

with

$$\Lambda_3 \equiv \frac{1}{\gamma \tau_v}, \tag{A.2}$$

and

$$p_3 = \bar{v} + \Lambda_3 \theta_3. \tag{A.3}$$

In the second period, we have

$$\begin{aligned} X_2(p_2) &= \gamma \frac{E_2[p_3] - p_2}{\text{Var}_2[p_3]} \\ &= -\Lambda_2^{-1}(p_2 - \bar{v}), \end{aligned} \tag{A.4}$$

with

$$\Lambda_2 \equiv \frac{1+\gamma\beta\Lambda_3^{-1}\tau_u}{\gamma\Lambda_3^{-2}\tau_u}, \tag{A.5}$$

and

$$p_2 = \bar{v} + \Lambda_2\theta_2. \tag{A.6}$$

Similar calculations show that in the first period $x_1 = -\Lambda_1^{-1}(p_1 - \bar{v})$ and $p_1 = \bar{v} + \Lambda_1\theta_1$, with

$$\Lambda_1 \equiv \frac{1+\gamma\beta\Lambda_2^{-1}\tau_u}{\gamma\Lambda_2^{-2}\tau_u}.$$

□

Proof of Corollary 1

For the first part, in the third period $\Lambda_3 = 1/\gamma\tau_v$, implying that $\partial\Lambda_3/\partial\beta = 0$. In the second period, using (A.5), $\partial\Lambda_2/\partial\beta = \Lambda_3 > 0$. Finally, in the first period we have

$$\Lambda_1 = \frac{\Lambda_2^2}{\gamma\tau_u} + \beta\Lambda_2,$$

which is increasing in β.

For the second part, we start from $\text{Cov}[p_3 - p_2, p_2 - p_1]$. Using (3) we have

$$\text{Cov}[p_3 - p_2, p_2 - p_1] = \frac{\text{Var}_2[p_3]}{\gamma}\frac{\Lambda_2}{\tau_u}\left(\frac{\beta\Lambda_2}{\gamma\tau_u} - 1\right),$$

which is positive if and only if

$$\frac{\beta\Lambda_2}{\gamma\tau_u} - 1 > 0.$$

Using the expression for Λ_2, this is equivalent to

$$\frac{\beta^2}{\gamma^2\tau_u\tau_v} + \frac{\beta}{(\gamma^2\tau_u\tau_v)^2} - 1 > 0,$$

which is satisfied provided that

$$\beta > \hat{\beta} \equiv \frac{-(\gamma^2\tau_u\tau_v)^{-1} + ((\gamma^2\tau_u\tau_v)^{-2} + 4(\gamma^2\tau_u\tau_v)^{-1})^{1/2}}{2(\gamma^2\tau_u\tau_v)^{-1}}.$$

Finally, for the right hand side of the above condition to be smaller than one, we need to make sure that

$$\gamma^2\tau_u\tau_v < \frac{2}{\sqrt{5}-1}.$$

Next, we compute $\mathrm{Cov}[v - p_3, p_3 - p_2]$. Using (3) we have

$$\mathrm{Cov}[v - p_3, p_3 - p_2] = \frac{\Lambda_3}{\tau_u}\left(\beta(1+\beta^2)\frac{\mathrm{Var}_2[p_3]}{\gamma} - \Lambda_3\right),$$

which is positive if and only if,

$$\beta(1+\beta^2)\frac{\Lambda_3}{\gamma\tau_u} - 1 > 0.$$

Using the expression for Λ_3, this is equivalent to

$$\beta^3 + \beta - \gamma^2\tau_u\tau_v > 0,$$

a condition that can be satisfied for β high enough, provided $\gamma^2\tau_u\tau_v < 2$. Therefore, we can conclude that if $\gamma^2\tau_u\tau_v < 2/(\sqrt{5}-1)$, for β sufficiently high short term returns display momentum in both the second and third period.

Computing the autocovariance for the long term return, yields

$$\mathrm{Cov}[v - p_3, p_1 - \bar{v}] = -\beta^2\Lambda_3\frac{\Lambda_1}{\tau_u}.$$

□

The following lemma establishes that working with the sequence $z^n \equiv \{z_t\}_{t=1}^n$ is equivalent to working with $p^n \equiv \{p_t\}_{t=1}^n$:

Lemma 3. *In any linear equilibrium the sequence of informational additions z^n is observationally equivalent to p^n.*

Proof. Consider a candidate equilibrium in linear strategies $x_{in} = a_n s_{in} - \varphi_n(p^n)$. In the first period imposing market clearing yields $\int_0^1 a_1 s_{i1} - \varphi_1(p_1)di + \theta_1 = a_1 v - \varphi_1(p_1) + \theta_1 = 0$ or, denoting with $z_1 = a_1 v + \theta_1$ the informational content of the first period order-flow, $z_1 = \varphi_1(p_1)$, where $\varphi_1(\cdot)$ is a linear function. Hence, z_1 and p_1 are observationally equivalent. Suppose now that $z^{n-1} = \{z_1, z_2, \ldots, z_{n-1}\}$ and $p^{n-1} = \{p_1, p_2, \ldots, p_{n-1}\}$ are observationally equivalent and consider the n-th period market clearing condition: $\int_0^1 X_n(s_{in}, p^{n-1}, p_n)di + \theta_n = 0$. Adding and subtracting $\sum_{t=1}^{n-1} \beta^{n-t+1} a_t v$, the latter condition can be rewritten as follows:

$$\sum_{t=1}^{n} z_t - \varphi_n(p^n) = 0,$$

where $\varphi_n(\cdot)$ is a linear function, $z_t = \Delta a_t v + u_t$ denotes the informational content of the t-th period order-flow, and $\Delta a_t = a_t - \beta a_{t-1},$. As by assumption p^{n-1} and z^{n-1} are observationally equivalent, it follows that observing p_n is equivalent to observing z_n. □

Proof of Proposition 2

To prove our argument, we proceed by backwards induction. In the last trading period traders act as in a static model and owing to CARA and normality we have

$$X_3(s_{i3}, z^3) = \gamma \frac{E_{i3}[v] - p_3}{\mathrm{Var}_{i3}[v]}, \tag{A.7}$$

and

$$p_3 = \alpha_{P_3}\left(v + \frac{\theta_3}{a_3}\right) + (1 - \alpha_{P_3})\, E_3[v], \tag{A.8}$$

where

$$a_3 = \gamma\tau_\epsilon \tag{A.9}$$

$$\alpha_{P_3} = \frac{\tau_\epsilon}{\tau_{i3}}, \tag{A.10}$$

and

$$\tau_{i3} \equiv \frac{1}{\text{Var}\left[v|s_{i3}, z^3\right]} = \tau_v + \tau_u \sum_{t=1}^{3} (\Delta a_t)^2 + \tau_\epsilon.$$

An alternative way of writing the third period equilibrium price is

$$p_3 = \lambda_3 z_3 + (1 - \lambda_3 \Delta a_3)\hat{p}_2, \tag{A.11}$$

where

$$\lambda_3 = \alpha_{P_3}\frac{1}{a_3} + (1 - \alpha_{P_3})\,\frac{\Delta a_3 \tau_u}{\tau_3}, \tag{A.12}$$

captures the price impact of the net informational addition contained in the 3rd period aggregate demand, while

$$\begin{aligned} \hat{p}_2 &= \frac{\alpha_{P_3}\tau_3\beta(\sum_{t=1}^{2}\beta^{2-t}z_t) + (1-\alpha_{P_3})a_3\tau_2 E_2[v]}{\alpha_{P_3}\tau_3\beta a_2 + (1-\alpha_{P_3})a_3\tau_2} \\ &= \frac{\gamma\tau_2 E_2[v] + \beta(z_2 + \beta z_1)}{\gamma\tau_2 + \beta a_2}, \end{aligned} \tag{A.13}$$

$\tau_n \equiv (\text{Var}[v|z^n])^{-1} = \tau_v + \tau_u \sum_{t=1}^{n}(\Delta a_t)^2$, $z_n = \Delta a_n v + u_n$, and $\Delta a_n = a_n - \beta a_{n-1}$.

Second Period

In the second period owing to CARA and normality, an agent i trades according to

$$X_2(s_{i2}, z^2) = \frac{\gamma(E_{i2}[p_3] - p_2)}{\text{Var}_{i2}[p_3]}, \tag{A.14}$$

where

$$E_{i2}[p_3] = \lambda_3 \Delta a_3 E_{i2}[v] + (1 - \lambda_3 \Delta a_3)\hat{p}_2 \tag{A.15}$$

$$\text{Var}_{i2}[p_3] = \lambda_3^2 \left(\frac{\tau_{i3}}{\tau_{i2}\tau_u}\right). \tag{A.16}$$

Replacing (A.15) and (A.16) in (A.14) yields

$$X_2(s_{i2}, z^2) = \frac{\gamma \Delta a_3 \tau_{i2} \tau_u}{\lambda_3 \tau_{i3}}(E_{i2}[v] - \hat{p}_2) + \frac{\gamma \tau_{i2} \tau_u}{\lambda_3^2 \tau_{i3}}(\hat{p}_2 - p_2).$$

Imposing market clearing and identifying equilibrium parameters yields

$$p_2 = \alpha_{P_2}\left(v + \frac{\theta_2}{a_2}\right) + (1 - \alpha_{P_2})E_2[v], \tag{A.17}$$

where

$$\alpha_{P_2} \equiv \alpha_{E_2}\left(1 + \frac{(\beta\rho_2 - 1)\tau_2}{\tau_{i3}}\right) \tag{A.18}$$

$$a_2 = \frac{\gamma \Delta a_3 \tau_u \tau_\epsilon}{\lambda_3 \tau_{i3}}, \tag{A.19}$$

and $\rho_2 \equiv a_2/(\gamma \sum_{t=1}^{2} \tau_{\epsilon t})$. Alternatively, in the spirit of what done for the third period analysis

$$p_2 = \lambda_2 z_2 + (1 - \lambda_2 \Delta a_2)\hat{p}_1, \tag{A.20}$$

where

$$\lambda_2 = \alpha_{P_2}\frac{1}{a_2} + (1-\alpha_{P_2})\frac{\Delta a_2 \tau_u}{\tau_2}, \tag{A.21}$$

and

$$\hat{p}_1 = \frac{\alpha_{P_2}\tau_2\beta z_1 + (1-\alpha_{P_2})a_2\tau_1 E_1[v]}{\alpha_{P_2}\tau_2\beta a_1 + (1-\alpha_{P_2})a_2\tau_1}. \tag{A.22}$$

Finally, note that using (A.18) and (A.19) and rearranging the expression for the second period strategy yields

$$X_2(s_{i2}, z^2) = \frac{a_2}{\alpha_{E_2}}(E_{i2}[v] - p_2) + \frac{\alpha_{P_2} - \alpha_{E_2}}{\alpha_{E_2}}\frac{a_2}{\alpha_{P_2}}(p_2 - E_2[v]).$$

First period

To compute the first period equilibrium we can use the results of the second period analysis. Indeed, $X_1(s_{i1}, z_1) = (\gamma/\mathrm{Var}_{i1}[p_2])(E_{i1}[p_2] - p_1)$ and using (A.20)

$$E_{i1}[p_2] = \lambda_2\Delta a_2 E_{i1}[v] + (1 - \lambda_2\Delta a_2)\hat{p}_1$$
$$\mathrm{Var}_{i1}[p_2] = \lambda_2^2\left(\frac{\tau_{i2}}{\tau_{i1}\tau_u}\right),$$

The above expressions highlight the recursive structure of the problem and imply

$$X_1(s_{i1}, z_1) = \frac{\gamma\Delta a_2\tau_{i1}\tau_u}{\lambda_2\tau_{i2}}(E_{i1}[v] - \hat{p}_1) + \frac{\gamma\tau_{i1}\tau_u}{\lambda_2^2\tau_{i2}}(\hat{p}_1 - p_1),$$

and

$$p_1 = \alpha_{P_1}\left(v + \frac{\theta_1}{a_1}\right) + (1-\alpha_{P_1})E_1[v], \tag{A.23}$$

where

$$\alpha_{P_1} = \alpha_{E_1}\left(1 + \tau_1\left(\frac{\gamma\alpha_{P_2}(\beta\rho_1 - \rho_2)}{a_2} + \frac{(\beta\rho_2 - 1)}{\tau_{i3}}\right)\right) \tag{A.24}$$

$$a_1 = \frac{\gamma\Delta a_2\tau_u\tau_\epsilon}{\lambda_2\tau_{i2}}, \tag{A.25}$$

and $\rho_1 \equiv a_1/(\gamma\tau_{\epsilon_1})$. In the first period too we can express the price as

$$p_1 = \lambda_1 z_1 + (1 - \lambda_1 a_1)\bar{v}, \tag{A.26}$$

with

$$\lambda_1 = \alpha_{P_1}\frac{1}{a_1} + (1 - \alpha_{P_1})\frac{a_1\tau_u}{\tau_1},$$

and, using (A.24), obtain that

$$X_1(s_{i1}, z_1) = \frac{a_1}{\alpha_{E_1}}(E_{i1}[v] - p_1) + \frac{\alpha_{P_1} - \alpha_{E_1}}{\alpha_{E_1}}\frac{a_1}{\alpha_{P_1}}(p_1 - E_1[v]). \tag{A.27}$$

□

Proof of Corollary 2

The first part of the Corollary is proved in the paper. For the second part, we show how to obtain the expression for Λ_2, the argument for Λ_1 being similar. Imposing market clearing on (A.7) yields

$$p_3 = \bar{E}_3[v] + \frac{\text{Var}_{i3}[v + \delta]}{\gamma}\theta_3,$$

where $\bar{E}_3[v] \equiv \int_0^1 E_{i3}[v]di$. Similarly, due to short term horizons, the second period price is given by

$$
\begin{aligned}
p_2 &= \bar{E}_2[p_3] + \frac{\text{Var}_{i2}[p_3]}{\gamma}\theta_2 \\
&= \bar{E}_2\left[\bar{E}_3[v]\right] + \frac{\text{Var}_{i3}[v+\delta]}{\gamma}\beta\bar{E}_2\left[\theta_2\right] + \frac{\text{Var}_{i2}[p_3]}{\gamma}\theta_2 \\
&= \alpha_{P_2}v + (1-\alpha_{P_2})E_2[v] + \left(\beta\frac{\text{Var}_{i3}[v+\delta]}{\gamma} + \frac{\text{Var}_{i2}[p_3]}{\gamma}\right)\theta_2,
\end{aligned} \tag{A.28}
$$

where

$$\alpha_{P_2} = \bar{\alpha}_{E_2} + \frac{\text{Var}_{i3}[v+\delta]}{\gamma}\beta a_2(1-\alpha_{E_2}),$$

and $\bar{\alpha}_{E_2}$ is given by:

$$\bar{\alpha}_{E_2} = \alpha_{E_2}\left(1 - \frac{\tau_2}{\tau_3}(1-\alpha_{E_3})\right).$$

Now, we know that at a linear equilibrium

$$p_2 = \alpha_{P_2}\left(v + \frac{\theta_2}{a_2}\right) + (1-\alpha_{P_2})E_2[v]. \tag{A.29}$$

Comparing (26) and (A.29), we then see that an alternative expression for a_2 is the following:

$$a_2 = \gamma\frac{\alpha_{P_2}}{\text{Var}_{i2}[p_3] + \beta\text{Var}_{i3}[v+\delta]}.$$

Given that we define the reciprocal of market depth in period 2 as $\Lambda_2 \equiv \alpha_{P_2}/a_2$, from the last equation we can conclude that

$$\Lambda_2 = \frac{\text{Var}_{i2}[p_3] + \beta\text{Var}_{i3}[v+\delta]}{\gamma}. \tag{A.30}$$

□

Proof of Corollary 3

To obtain equation (12) we start from the equilibrium price equation at time n:

$$p_n = \alpha_{P_n} \left(v + \frac{\theta_n}{a_n} \right) + (1 - \alpha_{P_n}) E_n[v],$$

and expanding the expression for θ_n, we obtain

$$p_n = \frac{\alpha_{P_n}}{a_n} (a_n v + u_n + \beta \theta_{n-1}) + (1 - \alpha_{P_n}) E_n[v].$$

Adding and subtracting $(\alpha_{P_n}/a_n)\beta a_{n-1} v$ at the r.h.s. of the above expression and rearranging

$$p_n = \lambda_n z_n + \frac{\beta \alpha_{P_n}}{a_n} (a_{n-1} v + \theta_{n-1}) + (1 - \alpha_{P_n}) \frac{\tau_{n-1}}{\tau_n} E_{n-1}[v], \tag{A.31}$$

where $z_n \equiv \Delta a_n v + u_n = (a_n - \beta a_{n-1}) v + u_n$, and

$$\lambda_n = \alpha_{P_n} \frac{1}{a_n} + (1 - \alpha_{P_n}) \frac{\Delta a_n \tau_u}{\tau_n}.$$

Adding and subtracting $(1 - \lambda_n \Delta a_n) p_{n-1}$ to the r.h.s. of (A.31) yields

$$\begin{aligned} p_n &= \lambda_n z_n + (1 - \lambda_n \Delta a_n) p_{n-1} - (1 - \lambda_n \Delta a_n) \times \\ &\left(\frac{\alpha_{P_{n-1}}}{a_{n-1}} (a_{n-1} v + \theta_{n-1}) + (1 - \alpha_{P_{n-1}}) E_{n-1}[v] \right) + \frac{\beta \alpha_{P_n}}{a_n} (a_{n-1} v + \theta_{n-1}) + (1 - \alpha_{P_n}) \frac{\tau_{n-1}}{\tau_n} E_{n-1}[v] \\ &= \lambda_n z_n + (1 - \lambda_n \Delta a_n) p_{n-1} + \left(\frac{\beta \alpha_{P_n}}{a_n} - (1 - \lambda_n \Delta a_n) \frac{\alpha_{P_{n-1}}}{a_{n-1}} \right) E_{n-1}[\theta_{n-1}]. \end{aligned} \tag{A.32}$$

We now prove that

$$\frac{\beta \alpha_{P_n}}{a_n} - (1 - \lambda_n \Delta a_n) \frac{\alpha_{P_{n-1}}}{a_{n-1}} = \lambda_n \Delta a_n \frac{\alpha_{P_{n-1}} - \alpha_{E_{n-1}}}{a_{n-1}}.$$

Starting from the third period price we have

$$
\begin{aligned}
p_3 &= \alpha_{P_3}\left(v + \frac{\theta_3}{a_3}\right) + (1-\alpha_{P_3})\,E_3[v] \qquad \text{(A.33)}\\
&= \lambda_3 z_3 + (1-\lambda_3\Delta a_3)\,p_2 + \left(\frac{\beta\alpha_{P_3}}{a_3} - (1-\lambda_3\Delta a_3)\frac{\alpha_{P_2}}{a_2}\right)E_2[\theta_2].
\end{aligned}
$$

Now using the definition for α_{P_3} we obtain

$$
\begin{aligned}
\frac{\beta\alpha_{P_3}}{a_3} - (1-\lambda_3\Delta a_3)\frac{\alpha_{P_2}}{a_2} &= \frac{\beta}{\gamma\tau_{i3}} - \frac{\gamma\tau_2+\beta a_2}{\gamma\tau_{i3}}\frac{\alpha_{P_2}}{a_2}\\
&= \frac{\beta a_2+\gamma\tau_2}{\gamma\tau_{i3}}\left(\frac{\beta}{\beta a_2+\gamma\tau_2} - \frac{\alpha_{P_2}}{a_2}\right)\\
&= \frac{(\beta a_2+\gamma\tau_2)}{\gamma\tau_{i3}}\left(\frac{\beta}{\beta a_2+\gamma\tau_2} - \frac{1}{\gamma\rho_2\tau_{i2}}\left(1+\frac{(\beta\rho_2-1)\tau_2}{\tau_{i3}}\right)\right)\\
&= \frac{1}{\gamma^2\rho_2\tau_{i2}\tau_{i3}}\left(\gamma\tau_1(\beta\rho_2-1) - (\beta a_2+\gamma\tau_2)\frac{(\beta\rho_2-1)\tau_2}{\tau_{i3}}\right)\\
&= \frac{(\beta\rho_2-1)\tau_2}{\gamma^2\rho_2\tau_{i2}\tau_{i3}^2}\left(\gamma\tau_{i3}-\beta a_2-\gamma\tau_2\right)\\
&= \frac{(\beta\rho_2-1)\tau_2}{\gamma\rho_2\tau_{i2}\tau_{i3}}\lambda_3\Delta a_3\\
&= \lambda_3\Delta a_3\frac{\alpha_{P_2}-\alpha_{E_2}}{a_2},
\end{aligned}
$$

where we use the definition of α_{P_2} to move from the second to the third row of the above expression. Summarizing, using the above result the second period price can be expressed as follows:

$$
p_3 = \lambda_3 z_3 + (1-\lambda_3\Delta a_3)\,p_2 + \lambda_3\Delta a_3\frac{\alpha_{P_2}-\alpha_{E_2}}{a_2}E_2[\theta_2]. \qquad \text{(A.34)}
$$

Going back one period, we need to prove that

$$
\frac{\beta\alpha_{P_2}}{a_2} - (1-\lambda_2\Delta a_2)\frac{\alpha_{P_1}}{a_1} = \lambda_2\Delta a_2\frac{\alpha_{P_1}-\alpha_{E_1}}{a_1}. \qquad \text{(A.35)}
$$

To prove this, we start by noting that

$$\frac{\beta\alpha_{P_2}}{a_2} - (1-\lambda_2\Delta a_2)\frac{\alpha_{P_1}}{a_1} = \frac{1}{a_1}\left(\alpha_{P_1}\lambda_2\Delta a_2 + \frac{\beta\alpha_{P_2}}{a_2}a_1 - \alpha_{P_1}\right),$$

which in turn implies that (A.35) is correct if and only if

$$\alpha_{P_1} = \alpha_{E_1}\lambda_2\Delta a_2 + \frac{\beta\alpha_{P_2}}{a_2}a_1. \tag{A.36}$$

To prove that the above expression is correct, we manipulate the definition of α_{P_1} to obtain:

$$\begin{aligned}
\alpha_{P_1} &= \alpha_{E_1}\left(1+\gamma\tau_1\left(\frac{\alpha_{P_2}(\beta\rho_2-\rho_1)}{a_2} + \frac{\alpha_{P_3}(\beta\rho_2-1)}{a_3}\right)\right)\\
&= \alpha_{E_1}\left(1+\tau_1\left(\frac{\alpha_{P_2}}{a_2}\left(\frac{\beta a_1}{\tau_{\epsilon_1}} - \frac{a_2}{\sum_{t=1}^2\tau_{\epsilon_t}}\right) + \frac{\beta\rho_2-1}{\tau_{i3}}\right)\right)\\
&= \frac{\alpha_{P_2}}{a_2}\beta a_1 + \alpha_{E_1}\left(1 - \frac{\alpha_{P_2}}{a_2}\beta a_1 + \left(\frac{\beta\rho_1-1}{\tau_{i3}} - \frac{\alpha_{P_2}}{\sum_{t=1}^2\tau_{\epsilon_t}}\right)\right).
\end{aligned} \tag{A.37}$$

Thus, to prove our claim we need to show that

$$1 - \frac{\alpha_{P_2}}{a_2}\beta a_1 + \left(\frac{\beta\rho_1-1}{\tau_{i3}} - \frac{\alpha_{P_2}}{\sum_{t=1}^2\tau_{\epsilon_t}}\right) = \lambda_2\Delta a_2. \tag{A.38}$$

According to our definition of α_{P_2} we have

$$\alpha_{P_2} = \alpha_{E_2}\left(1 + \frac{(\beta\rho_2-1)\tau_2}{\tau_{i3}}\right),$$

which in turn implies that

$$1-\alpha_{P_2} = (1-\alpha_{E_2})\left(1 - \frac{\sum_{t=1}^2\tau_{\epsilon_t}}{\tau_{i3}}(\beta\rho_2-1)\right). \tag{A.39}$$

Rearranging (A.38) yields

$$
\begin{aligned}
1-\frac{\alpha_{P_2}}{a_2}\beta a_1+\left(\frac{\beta\rho_1-1}{\tau_{i3}}-\frac{\alpha_{P_2}}{\sum_{t=1}^2\tau_{\epsilon_t}}\right) &= \frac{\alpha_{P_2}}{a_2}\Delta a_2+(1-\alpha_{P_2})-\frac{\tau_1}{\sum_{t=1}^2\tau_{\epsilon_t}}\left(\alpha_{P_2}-\frac{\sum_{t=1}^2\tau_{\epsilon_t}(\beta\rho_2-1)}{\tau_{i3}}\right)\\
&= \frac{\alpha_{P_2}}{a_2}\Delta a_2+(1-\alpha_{P_2})-\frac{\tau_1}{\tau_2}(1-\alpha_{P_2})\\
&= \lambda_2\Delta a_2, \qquad \text{(A.40)}
\end{aligned}
$$

where we use (A.39) to simplify the first row of the above expression. This allows us to write

$$
p_2=\lambda_2 z_2+(1-\lambda_2\Delta a_2)p_1+\lambda_2\Delta a_2\frac{\alpha_{P_1}-\alpha_{E_1}}{a_1}E_1[\theta_1], \qquad \text{(A.41)}
$$

and completes our proof. □

Proof of Proposition 4

Rearranging (12) yields

$$
\begin{aligned}
p_{n+1}-p_n &= \lambda_{n+1}\left(z_{n+1}+\Delta a_{n+1}\frac{\alpha_{P_n}-\alpha_{E_n}}{a_n}E_n[\theta_n]-\Delta a_{n+1}p_n\right)\\
&= \lambda_{n+1}\Delta a_{n+1}\left((1-\alpha_{E_n})(v-E_n[v])-\frac{\alpha_{E_n}}{a_n}\theta_n\right)+\lambda_{n+1}u_{n+1},
\end{aligned}
$$

and recalling that

$$
a_n=\gamma\frac{\lambda_{n+1}\Delta a_{n+1}\alpha_{E_n}}{\mathrm{Var}_{in}[p_{n+1}]},
$$

we can see that from the point of view of period n investors

$$
\begin{aligned}
E_n[p_{n+1}-p_n] &= -\frac{\lambda_{n+1}\Delta a_{n+1}\alpha_{E_n}}{a_n}E_n[\theta_n]\\
&= -\frac{\mathrm{Var}_{in}[p_{n+1}]}{\gamma}E_n[\theta_n]. \qquad \text{(A.42)}
\end{aligned}
$$

□

Proof of Corollary 5

The expression for rational investors' strategies are obtained in the proof of Proposition 2, whereas those for α_{P_2} and α_{P_1} follow immediately after rearranging (A.18) and (A.24). The recursive equation defining the couple a_1, a_2 follows from (A.19) and (A.25).

□

Proof of Proposition 3

For any $\beta \in [0,1]$, in the second period an equilibrium must satisfy $a_2 = \gamma\tau_\epsilon$. In the first period an equilibrium must satisfy

$$\begin{aligned}\phi_1(a_1, a_2) &\equiv a_1\lambda_2(\tau_2 + \tau_\epsilon) - \gamma\tau_\epsilon\Delta a_2\tau_u \\ &= a_1(1 + \gamma\tau_u\Delta a_2) - \gamma^2\tau_\epsilon\Delta a_2\tau_u = 0. \end{aligned} \tag{A.43}$$

The above equation is a quadratic in a_1 which for any $a_2 > 0$ and $\beta > 0$ possesses two positive, real solutions:

$$a_1^* = \frac{1 + \gamma\tau_u a_2(1+\beta) - \sqrt{(1 + \gamma\tau_u a_2(1+\beta))^2 - 4\beta(\gamma\tau_u a_2)^2}}{2\beta\gamma\tau_u} \tag{A.44}$$

$$a_1^{**} = \frac{1 + \gamma\tau_u a_2(1+\beta) + \sqrt{(1 + \gamma\tau_u a_2(1+\beta))^2 - 4\beta(\gamma\tau_u a_2)^2}}{2\beta\gamma\tau_u}, \tag{A.45}$$

with $a_1^{**} > a_1^*$. This proves that for $\beta > 0$ there are two linear equilibria.

Inspection of the above expressions for a_1 shows that $\beta a_1^* < a_2$, while $\beta a_1^{**} > a_2$. This implies that $\beta\rho_1 > 1$ for $a_1 = a_1^{**}$ and $\beta\rho_1 < 1$ otherwise. Thus, using (18), we obtain $\alpha_{P_1}(a_1^*, a_2) < \alpha_{E_1}(a_1^*, a_2)$, and $\alpha_{P_1}(a_1^{**}, a_2) > \alpha_{E_1}(a_1^{**}, a_2)$. The result for second period illiquidity follows from substituting (A.44) and (A.45) in λ_2. To see that prices are more informative along the low illiquidity equilibrium note that in the first period

$\text{Var}[v|z_1]^{-1} = \tau_1 = \tau_v + a_1^2\tau_u$. In the second period, the price along the low illiquidity equilibrium is more informative than along the high illiquidity equilibrium if and only if

$$\frac{(1+\beta^2+\gamma a_2\tau_u((1-\beta^2)+\beta(1+\beta^2)))\sqrt{(1+\gamma a_2\tau_u(1+\beta))^2-4\beta(\gamma a_2\tau_u)^2}}{\gamma^2\beta^2\tau_u} > 0,$$

which is always true.

To see that with $\beta = 0$, a unique equilibrium arises, note that

$$\lim_{\beta\to 0}\frac{1+\gamma\tau_u(a_2+\beta\gamma\tau_\epsilon)+\sqrt{1+\gamma\tau_u(2(a_2+\beta\gamma\tau_\epsilon)+\gamma\tau_u(a_2-\beta\gamma\tau_\epsilon)^2)}}{2\beta\gamma\tau_u} = \infty,$$

while, using l'Hospital's rule,

$$\lim_{\beta\to 0}\frac{1+\gamma\tau_u(a_2+\beta\gamma\tau_\epsilon)-\sqrt{1+\gamma\tau_u(2(a_2+\beta\gamma\tau_\epsilon)+\gamma\tau_u(a_2-\beta\gamma\tau_\epsilon)^2)}}{2\beta\gamma\tau_u} = \frac{\gamma a_2^2\tau_u}{1+\gamma a_2\tau_u}.$$

From (18) it then follows that in this case $\alpha_{P_1} < \alpha_{E_1}$. Finally, taking the limit of λ_2 as $\beta \to 0$ when $a_1 = a_1^*$ yields

$$\lim_{\beta\to 0}\lambda_2(a_1^*, a_2) = \frac{1+\gamma\tau_u a_2}{\gamma(\tau_v+(a_1^*)^2\tau_u+a_2^2\tau_u+\tau_\epsilon)} > 0,$$

whereas $\lim_{\beta\to 0}\lambda_2(a_1^{**}, a_2) = 0$. □

Proof of Lemma 2

Note that at any linear equilibrium $E_n[\theta_n] = a_n(v - E_n[v]) + \theta_n$. This, in turn, allows us to express the equilibrium price as follows:

$$p_n = \bar{E}_n[v] + \frac{\alpha_{P_n}-\alpha_{E_n}}{a_n}E_n[\theta_n] + \frac{\alpha_{E_n}}{a_n}\theta_n.$$

As a consequence

$$p_n - v = \bar{E}_n[v] - v + \frac{\alpha_{P_n} - \alpha_{E_n}}{a_n} E_n[\theta_n] + \frac{\alpha_{E_n}}{a_n}\theta_n.$$

Thus, if $\alpha_{P_n} > \alpha_{E_n}$ the price is closer to the fundamentals compared the consensus opinion, while the opposite occurs whenever $\alpha_{P_n} < \alpha_{E_n}$.

We now prove that at equilibrium (34) and (35) are equivalent. To see this, note that using (8) the covariance between p_n and v is given by

$$\text{Cov}[v, p_n] = \alpha_{P_n}\frac{1}{\tau_v} + (1 - \alpha_{P_n})\left(\frac{1}{\tau_v} - \frac{1}{\tau_n}\right), \tag{A.46}$$

and carrying out a similar computation for the time n consensus opinion

$$\text{Cov}\left[\bar{E}_n[v], v\right] = \alpha_{E_n}\frac{1}{\tau_v} + (1 - \alpha_{E_n})\left(\frac{1}{\tau_v} - \frac{1}{\tau_n}\right), \tag{A.47}$$

where $\tau_n \equiv \text{Var}[v|p^n] = \tau_v + \tau_u \sum_{t=1}^{n} \Delta a_t^2$. We can now subtract (A.47) from (A.46) and obtain

$$\text{Cov}\left[p_n - \bar{E}_n[v], v\right] = \frac{\alpha_{P_n} - \alpha_{E_n}}{\tau_n}, \tag{A.48}$$

implying that the price at time n over relies on public information if and only if the covariance between the price and the fundamentals falls short of that between the consensus opinion and the fundamentals. □

Proof of Corollary 7

To compute $\text{Cov}[p_2 - p_1, p_1 - \bar{v}]$ we rearrange expression (12) in the paper to obtain

$$\begin{aligned} p_2 - p_1 &= \lambda_2 u_2 + \lambda_2 \Delta a_2 \left((1 - \alpha_{E_1})(v - E_1[v]) - \frac{\alpha_{E_1}}{a_1}\theta_1\right) \\ &= \lambda_2 u_2 + \lambda_2 \Delta a_2 \left(\frac{\tau_v}{\tau_{i1}}(v - \bar{v}) - \frac{\tau_{i1} - \tau_v}{a_1 \tau_{i1}} u_1\right). \end{aligned} \tag{A.49}$$

Next, using expression (10) we obtain

$$\text{Cov}[p_2 - p_1, p_1 - \bar{v}] = -\frac{\lambda_1 \lambda_2 \Delta a_2 \tau_\epsilon}{a_1 \tau_u \tau_{i1}} < 0, \forall \beta \in (0, 1].$$

Computing the limit for $\beta \to 0$ of the above expression yields different results depending on whether we concentrate on the high or low illiquidity equilibrium. Indeed,

$$\lim_{\beta \to 0} \text{Cov}[p_2 - p_1, p_1 - \bar{v}]|_{a_1 = a_1^*} = -\lambda_1 \frac{(1 + \gamma \tau_u a_2)^2 \tau_\epsilon}{\gamma a_2 \tau_u^2 \tau_{i1}} < 0$$
$$\lim_{\beta \to 0} \text{Cov}[p_2 - p_1, p_1 - \bar{v}]|_{a_1 = a_1^{**}} = 0.$$

To compute the expression for $\text{Cov}[v - p_2, p_1 - \bar{v}]$ we use (A.50) and (10), and obtain

$$\text{Cov}[v - p_2, p_1 - \bar{v}] = -\frac{\beta \lambda_1}{\gamma \tau_{i2} \tau_u} < 0 \text{ for all } \beta > 0.$$

In this case the taking the limit of the above covariance for $\beta \to 0$ yields the same result across the two equilibria that arise:

$$\lim_{\beta \to 0} \text{Cov}[v - p_2, p_1 - \bar{v}]|_{a_1 = a_1^*} = \lim_{\beta \to 0} \text{Cov}[v - p_2, p_1 - \bar{v}]|_{a_1 = a_1^{**}} = 0.$$

To compute $\text{Cov}[v - p_2, p_2 - p_1]$ we use (A.49) and using again expression (10) we get

$$\begin{aligned} v - p_2 &= (1 - \alpha_{E_2})(v - E_2[v]) - \frac{\alpha_{P_2}}{a_2} \theta_2 \\ &= \frac{\tau_v (v - \bar{v})}{\tau_{i2}} - \frac{\beta + \gamma \tau_u a_1}{\gamma \tau_{i2}} u_1 - \frac{1 + \gamma \tau_u \Delta a_2}{\gamma \tau_{i2}} u_2. \end{aligned} \tag{A.50}$$

Using (A.49) and (A.50) we can now compute the autocovariance of returns and get

$$\begin{aligned}\text{Cov}[v - p_2, p_2 - p_1] &= \lambda_2 \left(\frac{\Delta a_2 \tau_v}{\tau_{i1}\tau_{i2}} - \frac{1 + \gamma\tau_u \Delta a_2}{\gamma\tau_{i2}\tau_u} + \Delta a_2 \frac{\tau_{i1} - \tau_v}{a_1 \tau_{i1}} \frac{\beta + \gamma\tau_u a_1}{\gamma\tau_{i2}\tau_u} \right) \\ &= -\frac{\lambda_2}{\gamma\tau_{i2}\tau_u} \left(1 - \beta\Delta a_2 \frac{\tau_{i1} - \tau_v}{a_1\tau_{i1}} \right). \qquad \text{(A.51)}\end{aligned}$$

Looking at (A.51) we can immediately say that along the equilibrium with low illiquidity there is momentum. This is true because in that equilibrium $\lambda_2 < 0$ and $\Delta a_2 < 0$. Along the equilibrium with high illiquidity momentum can occur, *depending on the persistence of liquidity trades.* To see this, note that since in this equilibrium $\lambda_2 > 0$ and $\Delta a_2 > 0$, from (A.51) momentum needs

$$1 - \beta\Delta a_2 \frac{\tau_{i1} - \tau_v}{a_1\tau_{i1}} < 0,$$

which can be rearranged as an (implicit) condition on the magnitude of β:

$$\frac{a_1\tau_{i1}}{\Delta a_2(\tau_{i1} - \tau_v)} < \beta < 1.$$

If $\beta = 0$, the above condition is never satisfied. Indeed, in this case there exists a unique equilibrium in which $\Delta a_2 = a_2 > 0$. Therefore, when $\beta = 0$ returns always display reversal. If $\beta = 1$, the condition is satisfied if

$$a_1\tau_v + a_1(\tau_\epsilon + a_1^2\tau_u) < \Delta a_2\tau_u(\tau_\epsilon + a_1^2\tau_u).$$

Isolating τ_v in the above expression yields:

$$\tau_v < \hat{\tau}_v \equiv \frac{(\Delta a_2 - a_1)(\tau_\epsilon + a_1^2\tau_u)}{a_1}, \qquad \text{(A.52)}$$

which, since a_1 does not depend on τ_v (see (A.44)), gives an explicit upper bound on τ_v.

Hence, if $\tau_v < \hat{\tau}_v$, there exists a $\hat{\beta}$ such that for all $\beta \geq \hat{\beta}$ momentum occurs between the second and third period returns along the equilibrium with high illiquidity. $\square$

Proof of Corollary 9

Suppose that $u_n \sim N(\bar{u}, \tau_u^{-1})$, with $\bar{u} > 0$. Then, it is easy to that that $\{z_1, z_2\}$ is observationally equivalent to $\{p_1, p_2\}$, while

$$a_1^{-1}(z_1 - \bar{u}) \equiv v + a_1^{-1}(u_1 - \bar{u})|v \sim N(v, a_1^{-2}\tau_u^{-1})$$
$$(\Delta a_2)^{-1}(z_2 - \bar{u}) \equiv v + (\Delta a_2)^{-1}(u_2 - \bar{u})|v \sim N(v, (\Delta a_2)^{-2}\tau_u^{-1}).$$

Therefore, nothing changes for the precisions in the projection expressions while

$$\begin{aligned} E_{in}[v] &= \frac{\overbrace{\tau_v\bar{v} + \tau_u \sum_{t=1}^{n} \Delta a_t(z_t - \bar{u})}^{\tau_2 E_2[v]} + \tau_\epsilon s_{in}}{\tau_{in}} \\ &= \frac{\tau_2 E_2[v] + \tau_\epsilon s_{in}}{\tau_{in}}. \end{aligned}$$

As a result, everything works as in the model with $\bar{u} = 0$, except that now there is a premium above $\bar{v}$ in the expression for the expected price, that is

$$E[p_n] = \bar{v} + \Lambda_n E[\theta_n]. \tag{A.53}$$

Using (A.53) we can now compute

$$E[v - p_2] = -\Lambda_2 E[\theta_2] = -\Lambda_2(1 + \beta)\bar{u}$$
$$E[p_1 - \bar{v}] = \Lambda_1 E[\theta_1] = \Lambda_1 \bar{u},$$

and using the fact that $0 = E[v - \bar{v}] = E[v - p_2 + p_2 - p_1 + p_1 - \bar{v}]$, obtain

$$E[p_2 - p_1] = (\Lambda_2(1+\beta) - \Lambda_1)\bar{u}.$$

□

Proof of Corollary 10

At time 1, owing to (11), we have

$$\Lambda_1 = \frac{\text{Var}_{i1}[p_2]}{\gamma} + \beta\Lambda_2.$$

Denote by Λ_2^H, and Λ_2^L, respectively the inventory related component of second period illiquidity along the equilibrium with high and low illiquidity. Given that along the high illiquidity equilibrium prices are more informative (see Proposition 3), we have

$$\Lambda_2^H \equiv \frac{1}{\gamma\tau_{i2}^H} > \Lambda_2^L \equiv \frac{1}{\gamma\tau_{i2}^L}. \tag{A.54}$$

This proves the last part of our result. Next, we need to prove that along the equilibrium with low illiquidity, price risk (captured by $\text{Var}_{i1}[p_2]$) is lower compared to the equilibrium with high illiquidity. To see this, note that according to (19)

$$a_1 = \gamma \frac{\lambda_2 \Delta a_2}{\text{Var}_{i1}[p_2]} \alpha_{E_1}.$$

Now, given that $a_2 = \gamma\tau_\epsilon$, by direct comparison one can verify that $\lambda_2(a_1^*, a_2)(a_2 - \beta a_1^*) > \lambda_2(a_1^{**}, a_2)(a_2 - \beta a_1^{**})$. Given that $a_1^{**} > a_1^*$ and $\alpha_{E_1}(a_1^*) > \alpha_{E_1}(a_1^{**})$, this implies that $\text{Var}_{i1}[p_2]$ must be lower along the equilibrium with low illiquidity compared to the other equilibrium. This, together with (A.54), implies that $\Lambda_1^H > \Lambda_1^L$.

□