A proposal for the design of energy-related scenario for stock stress testing

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Summary

This article proposes a flexible methodology that captures the asymmetry in the relationship between the stock market and the oil market jointly with potential structural changes. It deals with the challenge of modelling the sharp increase in dependence across markets in stress situations. The study analyses the response of the European stock market to an extreme energy-related scenario. This exercise is of particular significance given the growing interest in the consequences of energy prices for the real economy and the risks of a disruptive transition to a low-carbon economy.

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1 Introduction

The Paris agreements advocate a reduction in greenhouse gas emissions to facilitate an energy transition towards an economy led by renewable sources of energy. The European Commission adopted measures in this direction with the publication of Directive 2009/28/EC, of the European Parliament and of the Council, of 23 April 2009, on the promotion of the use of energy from renewable sources (the 'Renewable Energy Directive', or 'RED'), proposing a 70% reduction in fossil fuel use by 2050. These measures must be considered jointly with an analysis of the exposure of the main sectors of the economy to extreme movements in fossil fuel prices, so as to be able to monitor the transition and avoid a disruptive process that could harm to the economy. This article proposes an approach to design energy-related scenarios for stock test testing[1] purposes that would serve as a tool to identify potential spillovers, without overlooking the connections that may arise through other variables related to the economic cycle. The link between stock and oil market presents numerous channels of transmission. A great number of industries use oil-based products as production factors, such as kerosene and plastic materials. Increases in production costs could affect the performance of companies and their sales price, generating inflationary processes.

We need to take into account the features that emerge in the empirical data to avoid misleading conclusions when performing an econometric analysis. The literature identifies four main features in the joint dependence that must be consider to correctly reflect the empirical evidence: non-linearities, structural changes, asymmetric behaviours and tail dependence (*i.e.* the increased likelihood of having extreme observations in one market when extreme realizations occur in the other market).

The combination of a model of stochastic switching regimes, i.e. Markov switching models, with a copula methodology helps disclose hidden patterns in the data and sheds light on possible changes in tail dependence. To my knowledge, the combination of these approaches to deal with the particular features of the empirical data has not been yet employed in this topic.

The results of the Eurostoxx index and its subsectors over the period 2000-2015 record the maximum weekly losses, exceeded only twice per year, i.e. the 5-th percentile of the distribution of weekly returns. This article addresses the challenge of translating the impact of extreme scenarios in oil prices into a useful measure with functions for risk management, improving the financial education of investors. Conditional Value at Risk (*CoVaR*) provides a simple way of summarising this complex information by

1 It should be noted that the term stress test commonly indicates a range of potential analytical techniques and exercises.

measuring the exposure of a stock portfolio to abrupt movements in oil prices. The response of each productive sector to the same oil-related scenario would provide evidence about portfolio strategies when extreme swings in oil prices materialise. The results point to a structural change during the 2008 financial crisis, when the sign of the relationship between the two markets switched from negative to positive. The economic cycle and its implications for the profit margin, oil demand and the investors'' herd behaviour could explain this change in dependence. A reduction in the price of production factors, such as oil, increases the margin between the sale price and the unit production cost. However, an increase in the oil price typically entails a general increase in production costs, which would be passed on to sale prices during the expansive phase of the economic cycle. This would explain the lower dependence between oil and the European stock market on a scenario where oil prices are rising abruptly. The onset of the financial crisis led to losses for companies and a sharp increase in unemployment, resulting in a substitution effect between oil and employment (Fernández *et al.,* 2012)[2] and a downturn in the aggregate demand. This study also shows a reduction in the exposure of the stock portfolio to oil price shocks, through diversification across sectors that present a different elasticity in the demand of their products, such as the health care sector.

This research has implications for investors and portfolio managers, who need to evaluate the potential losses of their stock portfolio losses on oil-related scenarios, improving their hedging of commodity risk. It is also of interest for regulatory authorities, who need to supervise listed companies, measuring their exposure to oil price fluctuations. Lastly, policy makers can find in the proposed approach a way to measure possible risks arising from a disruptive transition towards a low-carbon economy, and the potential impact of extreme movements in oil prices on the economy.

The remainder of the article has the following structure: section 2 provides a brief overview of the literature; section 3 summarises the methodology, while section 4 presents the data used to conduct the empirical exercise from section 5. Lastly, section 6 sets out the main conclusions of this paper.

2 Fernández *et al.* (2012), *op. cit.*

2. Literature review

The relationship between oil and stock market has been extensively studied in the literature for several reasons. On the one hand, the effects of oil price shocks go beyond inflation and a downturn in corporate profits. An oil price shock may also be reflected in aggregate production and employment measurements (Hamilton, 1983; Mork, 1989; Hooker, 1999)[3], since the oil price is a benchmark price to establish energy prices. On the other hand, stock market performance could be interpreted as a high-frequency proxy of the economy allowing us to analyse the short-term impact of oil price shocks on the economy.

The economic and financial literature shows us that the relationship between oil stock market depends on the productive structure of a particular region (Arouri & Nguyen, 2010; Ramos & Veiga, 2013; Park & Ratti, 2008; Arouri, Jouni & Nguyen, 2011; Arouri, Jouni & Nguyen, 2012)[4]. In this regard, Lee, Yang & Huang (2012)[5] indicates that, depending on the sectoral diversification of each country, the analysis of domestic indices may mask the impact of an oil price shock on the economy . Because of this, the empirical exercise in this article is performed considering a sectoral division of the European stock market, rather than a country-based division.

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3 Hamilton, J.D. (1983). "Oil and the macroeconomy since World War II". *Journal of Political Economy*, Vol. 91, No. 2, pp. 228-248; Mork, K.A. (1989). "Oil and the macroeconomy when prices go up and down: An extension of Hamilton's results". *Journal of Political Economy*, Vol. 97, No. 3, pp. 40-744; Hooker, M.A. (1999). *Oil and the macroeconomy revisited*. FEDS Working Paper No. 99-43.

4 Arouri, M. & Nguyen, D. (2010). "Oil prices, stock markets and portfolio investment: Evidence from sector analysis in Europe over the last decade". *Energy Policy*, Vol. 38, No. 8, pp. 4.528-4.539; Ramos, S.B. & Veiga, H. (2013). "Oil price asymmetric effects: Answering the puzzle in international stock markets". *Energy Economics*, Vol. 38, pp. 136-145; Park, J. & Ratti, R. (2008). "Oil price shocks and stock markets in the U.S. and 13 European countries". Energy Economics, Vol. 30, No. 5, pp. 2,587-2,608; Arouri, M., Jouini, J. & Nguyen, D. (2011). "Volatility spillovers between oil prices and stock sector returns: Implications for portfolio management". *Journal of International Money and Finance*, Vol. 30, No. 7, pp. 1,387-1,405; Arouri, M., Jouini, J. & Nguyen, D. (2012). "On the impacts of oil price fluctuations on European equity markets: Volatility spillover and hedging effectiveness". *Energy Economics*, Vol. 34, No. 2, pp. 611-617.

5 Lee, B.J., Yang, C.W. & Huang, B.N. (2012). "Oil price movements and stock markets revisited: A case of sector stock price indexes in the G-7 countries". *Energy Economics*, Vol. 34, No. 5, pp. 1,284-1,300.

The presence of non-linearities refers to the changes in how the stock market and oil behave arising on extreme scenarios. Ciner (201)[6] indicates that ignoring this risk can serve to deny the impact of the movements in oil price on the stock market (Apergis & Miller, 2009; Chen, Roll & Ross, 1986; Huang, Masulis & Stoll, 1996)[7]. Reboredo (2010)[8] underlines this feature in his analysis, finding structural changes in dependence between oil and stock markets. This dependence presents also an asymmetric pattern. The variables reveal different behaviour depending on the scenario for the price trend (bullish or bearish scenario), which must be taken into account when designing a stress test. For instance, Aloui, Hammoudeh & Nguyen (2013)[9] show evidence of the increase in dependence between both markets during periods of financial distress. Lastly, tail dependence plays a key role in understanding the link between both markets within extreme scenarios, in particular when they experience downward pressure on prices (Aloui *et al.*, 2013; Nguyen & Bhatti, 2012; Wen, Wei & Huang, 2012)[10]. The investors' herd behaviour during the contraction phase of the economic cycle could help to explain this feature.

6 Ciner, C. Energy shocks and financial markets: nonlinear linkages. Studies in Nonlinear Dynamics & Econometrics, 5(3), 2001.. Link: https://pdfs.semanticscholar.org/2380/e7262912232ca478b31225b8eb317ffb1abb.pdf

7 Apergis, N. & Miller, S.M. (2009). "Do structural oil-market shocks affect stock prices?". *Energy Economics*, Vol. 31, No. 4, pp. 569-575; Chen, N.F., Roll, R. & Ross, S.A. (1986). "Economic forces and the stock market". *Journal of Business*, Vol. 59, No. 3, pp. 383-403; Huang, R., Masulis, R. & Stoll, H. (1996). "Energy shocks and financial markets". *Journal of Futures Markets: Futures, Options, and Other Derivative Products*, Vol. 16, No. 1, pp. 1-27.

8 Reboredo, J.C. (2010). "Nonlinear effects of oil shocks on stock returns: a Markov-switching approach". *Applied Economics*, Vol. 42, No. 29, pp. 3,735-3,744.

9 Aloui, R., Hammoudeh, S. & Nguyen, D.K. (2013). "A time-varying copula approach to oil and stock market dependence: The case of transition economies". *Energy Economics*, Vol. 39, No. C, pp. 208-221.

10 Aloui *et al.* (2013), *op. cit.*; Nguyen, C. & Bhatti, M. (2012). "Copula model dependency between oil prices and stock markets: Evidence from China and Vietnam". *Journal of International Financial Markets, Institutions and Money*, Vol. 22, No. 4, pp. 758-773; Wen, X., Wei, Y. & Huang, D. (2012). "Measuring contagion between energy market and stock market during financial crisis: A copula approach". *Energy Economics*, Vol. 34, No. 5, pp. 1,435-1,446.

We propose combining a copula methodology, simultaneously considering asymmetries, tail dependence and non-linearities, with a Markov switching model, identifying structural changes in data. This approach allows us to reflect all the joint features between stock market and oil markets shown by the financial data. The huge flexibility of the copula to gather different statistical features explains the growing interest shown over recent years in addressing the analysis of spillovers between the oil stock markets (Sukcharoen, Zohrabyan, Leatham & Wu, 2014; Nguyen & Bhatti, 2012; Wen *et al.*, 2012; Reboredo & Ugolini, 2016; Mensi, Hammoudeh, Shahzad & Shahbaz, 2017)[11]. The Markov switching methodology is a state-of-the-art approach that allows for identifying endogenously different regimes over time with a clear economic interpretation. This model has been used to capture regime changes between oil returns and economic variables such as: GDP growth (Raymond & Rich, 1997; Clements & Krolzig, 2002; Holmes & Wang, 2003; Manera & Cologni, 2006)[12], sectoral employment (Fernández, Pérez & Ruiz, 2012)[13] and stock market (Balcilar, Gupta & Miller, 2015; Reboredo, 2010; Aloui & Jammazi, 2009)[14].

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11 Sukcharoen, K., Zohrabyan, T., Leatham, D. & Wu, X.(2014). "Interdependence of oil prices and stock market indices: A copula approach". *Energy Economics*, Vol. 44, No. C, pp. 331-339; Nguyen & Bhatti (2012), *op. cit.*; Wen *et al.* (2012), *op. cit.*; Reboredo, J.C. & Ugolini, A. (2016). "Quantile dependence of oil price movements and stock returns". *Energy Economics*, Vol. 54, No. C, pp. 33-49; Mensi, W., Hammoudeh, S., Shahzad, S. & Shahbaz, M. (2017). "Modelling systemic risk and dependence structure between oil and stock markets using a variational mode decomposition-based copula method". *Journal of Banking & Finance*, Vol. 75, pp. 258-279.

12 Raymond, J. & Rich, R. (1997). "Oil and the macroeconomy: A Markov state-switching approach". *Journal of Money, Credit, and Banking*, Vol. 29, No. 2, pp. 193-213; Clements, M. & Krolzig, H. (2002). "Can oil shocks explain asymmetries in the US business cycle?". *Advances in Markov-Switching Models*, Springer, pp. 41-60; Holmes, M. & Wang, P. (2003). "Oil price shocks and the asymmetric adjustment of UK output: A Markov switching approach". *International Review of Applied Economics*, Vol. 17, No. 2, pp. 181-192; Manera, M. & Cologni, A. (2006). *The Asymmetric Effects of Oil Shocks on Output Growth: A Markov-Switching Analysis for the G-7 Countries.* Fondazione Eni Enrico Mattei, Nota di Lavoro No. 29.

13 Fernández, E., Pérez, R. & Ruiz, J. (2012). "Análisis dinámico del impacto de los shocks en el precio del petróleo sobre el empleo por sectores productivos". *Economía industrial*, Vol. 384, pp. 85-98.

14 Balcilar, M., Gupta, R. & Miller, S.M. (2015). "Regime switching model of US crude oil and stock market prices: 1859 to 2013". *Energy Economics*, Vol. 49, No. C, pp. 317-327; Reboredo (2010), *op. cit.*; Aloui, C. & Jammazi, R. (2009). "The effects of crude oil shocks on stock market shifts behaviour: A regime switching approach". *Energy Economics*, Vol. 31, No. 5, pp. 789-799.

3. Methodology

This section introduces the concept of Conditional Value Risk (*CoVaR*), which will be the main measure in the proposed stress test. A brief introduction about copula methodology will be follow by a short presentation on how it can be used to calculate the *CoVaR*. The section finally ends highlighting some modelling issues.

3.1 The concept of *CoVaR*

A certain percentile of a returns distribution conditional on a particular scenario, known as *CoVaR* (Adrian & Brunnermeier, 2016; Girardi & Ergün, 2013[15]), is the main indicator employed in this paper to measure the behaviour in the tail of the distribution when distress scenarios materialise. This measure provides a conditional view of the Value at Risk (*VaR*) measure, which is widely used for risk management purposes and setting capital requirements in the financial sector. The *CoVaR* translates those spillovers generated by the oil market into potential losses for stock market.

The difference between the maximum losses of a stock market on an oil-related distress scenario (hereinafter, the *CoVaR*) and the maximum losses of the stock market independently of the oil-related scenario (*VaR* provides a quantitative assessment on how much the risk may change for a stock portfolio when a distress scenario materialises in the oil market. This indicator is very useful for investors, because it helps them to be aware of the consequences of exposure to oil, and also for market authorities and policy makers, who need to monitor the magnitude of spillovers between these key markets.

The *CoVaR* measure focuses on the tail of the distribution of stock returns, where non-linearities and asymmetries appear, and where the effects of the scenarios considered are most harmful. The scenarios considered for the *CoVaR assessment* depend on the movement in oil prices. On the one hand, a bullish *CoVaR* is computed as the VaR of the stock returns when oil is above its highest $1-\alpha$ percentile. In mathematical notation:

15 Adrian, T. and Brunnermeier, M. K. CoVaR. American Economic Review, 106(7):1705-41, 2016. Link: https://www.aeaweb.org/articles?id=10.1257/aer.20120555. Girardi, G. and Ergun, A. T. Systemic risk measurement: Multivariate GARCH estimation of CoVaR. Journal of Banking and Finance, 37(8):3169{3180, 2013. Link: https://econpapers.repec.org/article/eeejbfina/v_3a37_3ay_3a2013_3ai_3a8_3ap_3a3169-3180.htm

$$P\big(r_o > VaR(1-\alpha)\big) = \alpha,$$

where $VaR(1-\alpha)$ indicates the percentile $(1-\alpha)100\%$ of the distribution of oil returns (r_o). On the other hand, the scenario for a bearish *CoVaR* is defined as that one where oil returns is below the α percentile, *i.e.*:

$$P\big(r_o < VaR(\alpha)\big) = \alpha.$$

When these scenarios materialise, the *CoVaR* defines a percentile β of the distribution of stock returns. In other words, the bearish *CoVaR* is implicitly obtained from:

$$P\big(r_m < CoVaR|\, r_o < VaR(\alpha)\big) = \frac{P\big(r_m < CoVaR, r_o < VaR(\alpha)\big)}{P\,\big(r_o < VaR(\alpha)\big)} = \beta, \tag{1}$$

where the equivalence between the left-hand and right-hand parts is obtained from the Bayes theorem[16]. Similarly, the bullish *CoVaR* is implicitly obtained from:

$$P\big(r_m < CoVaR|\, r_o > VaR(1-\alpha)\big) = \frac{P\big(r_m < CoVaR, r_o > VaR(1-\alpha)\big)}{P\,\big(r_o > VaR(1-\alpha)\big)} = \beta, \tag{2}$$

3.2 The copula methodology

The proposed approach allows for a direct of joint probability of oil and stock market returns, into the probability of observing these events separately, and a function that defines the relationship between the two events, i.e. the copula. The copula provides a high flexible model for the joint distribution, gathering different characteristics, such as asymmetric dependence or the existence of tail dependence. Sklar's theorem (1959)[17] establishes that any joint distribution can be expressed as a combination of marginal distribution and copulas, i.e.:

$$F(r_o, r_m) = C\big(F_o(r_o), F_m(r_m)\big), \tag{3}$$

where F_k is the marginal distribution function of a variable $k = o,m,$ where o is a sub-index to refer to the oil market, m refers to the variable income market, and $C(\dots)$ is a copula function.

Using this methodology, the numerator in equation (1) may be expressed as $C(F_m(CoVaR), \alpha)$, and consequently the *CoVaR* will express the value of the stock market returns such that the following equality holds:

16 The Bayes theorem indicates that the probability of observing an event B conditional on an event A can be defined as the joint probability of observing both events standardised by the probability of observing conditioning event A, *i.e.* . $P(B|A) = \frac{P(B,A)}{P(A)}$.

17 Sklar, M. (1959). "Fonctions de repartition à n dimensions et leurs marges". Publications de l'Institut Statistique de l'Université de Paris, Vol. 8, pp. 229-231

$$C(F_m(CoVaR), \alpha) - \alpha\beta = 0.$$

In the case of the bullish *CoVaR* in equation (2), similarly, the equality to be hold is:

$$F_m(CoVaR) - C(F_m(CoVaR), 1 - \alpha) - \alpha\beta = 0.$$

Once the parameters of the model have been estimated, we can obtain the *CoVaR* for certain values of α and β by using any optimization algorithm that could find where the above functions are equal to zero[18].

3.3 Marginal distribution and dependence structure

This subsection introduces the model for the marginal distributions in a first stage, and secondly the dependence structure defined by the copula. These two stages allow us to obtain the joint distribution of returns.

3.3.1 Marginal distribution

The distribution of the marginal density of oil returns (*o*) and stock market (*m*) is represented by a first order autoregressive model (AR (1))[19], *i.e.:*

$$r_{k,t} = \phi_{k,0} + \phi_{k,1} r_{k,t-1} + \epsilon_{k,t} \qquad k = o, m \tag{4}$$

where $\phi_{k,j}$ is the parameter of model AR (1) and $\epsilon_{k,t} = \sigma_{k,t} z_{k,t}$. The variance of the $\epsilon_{k,t}$ term follows a GJR-GARCH (1,1), reflecting a leverage effect[20], in other words:

$$\sigma^2_{k,t} = \omega_k + \beta_k \sigma^2_{k,t-1} + \left(\alpha_k + \gamma_k \mathbb{1}_{\epsilon_{k,t}<0}\right)\epsilon^2_{k,t-1}, \qquad k = o, m \tag{5}$$

where ω_k, β_k and α_k are the parameters of the GARCH model and $\mathbb{1}_{\epsilon_{k,t}<0}$ is an indicator function that values 1 $\epsilon_{k,t} < 0$ and 0 otherwise γ_k captures the leverage effect, in other words the fact that negative shocks have a greater impact on variance than positive ones. When $\gamma_k = 0$, we obtain the GARCH model. Lastly, $z_{k,t}$ is an independent and identically distributed random with zero mean and unit variance following an asymmetric Student t-distribution (Hansen, 1994)[21]. Its density function is:

18 In particular, the *fzero* function of the MATLAB software has been used in this study to find this value.

19 The autoregressive process is a regression model in which the explanatory variables are the same dependent variable lagged *p* times, where *p* indicates the number of lags, i.e. the order of the autoregressive model.

20 In a GARCH model, the variance today is explained by three related components: the variance of the previous period, the square of the non-predictable part of the autoregressive model, and a long-term variance component. The asymmetric behaviour of GJR-GARCH is reflected in the fact that negative news have a grater impact on market variance than positive news.

21 Hansen, B. E. (1994). "Autoregressive conditional density estimation". *International Economic Review*, Vol. 35, No. 3, pp. 705-730.

$$f(z_{k,t}|\eta_k,\lambda_k) = \begin{cases} bc\left(1+\dfrac{1}{\eta_k-2}\left(\dfrac{b\,z_{k,t}+a}{1-\lambda_k}\right)^2\right)^{-\frac{\eta_k+1}{2}} & si \quad z_{k,t} < -a/b \\ bc\left(1+\dfrac{1}{\eta_k-2}\left(\dfrac{b\,z_{k,t}+a}{1+\lambda_k}\right)^2\right)^{-\frac{\eta_k+1}{2}} & si \quad z_{k,t} \geq -a/b \end{cases} \tag{6}$$

where $2 < \eta_k < \infty$ and $-1 < \lambda_k < 1$. The constants a, b and c, are given by $a = 4c\lambda_k\left(\frac{\eta_k-2}{\eta_k-1}\right)$, $b = \sqrt{1+3\lambda_k^2-a^2}$ y $c = \frac{\Gamma\left(\frac{\eta_k+1}{2}\right)}{\sqrt{\pi(\eta_k-2)}\Gamma\left(\frac{\eta_k}{2}\right)}$. Note that when $\lambda_k = 0$, Equation (6) is reduced to the normal distribution as the number of degrees of freedom η_k tends to infinity. When $\lambda_k = 0$ and the number of degrees of freedom is finite, we obtain the standardised Student t-distribution.

3.3.2 Dependence structure

Five types of copula are initially considered to reflect the relationships among the variables, chosen for their different characteristics regarding tail dependence. The Gaussian and Student t copulas allow for a positive and negative association between variables. While the former does not exhibit tail dependence, the Student t copula presents symmetrical tail dependence. Moreover, the Gumbel and Clayton copulas allow only for positive asymmetrical associations. The Clayton copula has lower tail dependence, while the Gumbel copula presents upper tail dependence. Lastly, the BB1 copula, also known as the Clayton-Gumbel copula, allows only for positive association, but this may be asymmetrical. Table 1 shows the features of these copulas in terms of tail dependence.

Main tail dependence characteristics for each copula TABLE 1

Copula	Lower tail dependence	Upper tail dependence
Gaussian	–	–
Student t	$2\, t_{\eta+1}\left(-\sqrt{\frac{(\eta+1)(1-\rho)}{1+\rho}}\right)$	$2\, t_{\eta+1}\left(-\sqrt{\frac{(\eta+1)(1-\rho)}{1+\rho}}\right)$
Clayton	$2^{-\frac{1}{\theta}}$	–
Gumbel	–	$2-2^{\frac{1}{\theta}}$
BB1	$2^{-\frac{1}{\delta\theta}}$	$2-2^{\frac{1}{\delta}}$

Source: Ao, Kim & Amouzegar (2017); Jiang (2012), Joe & Hu (1996), Fischer (2003) & Joe (1997)[22]. Note: – represents lack of tail dependence. ρ and ν are the correlation parameters and the number of degrees of freedom of the Student t copula. θ is the dependence parameter of the Clayton and Gumbel copula, while θ and δ are the two parameters of the BB1 copula.

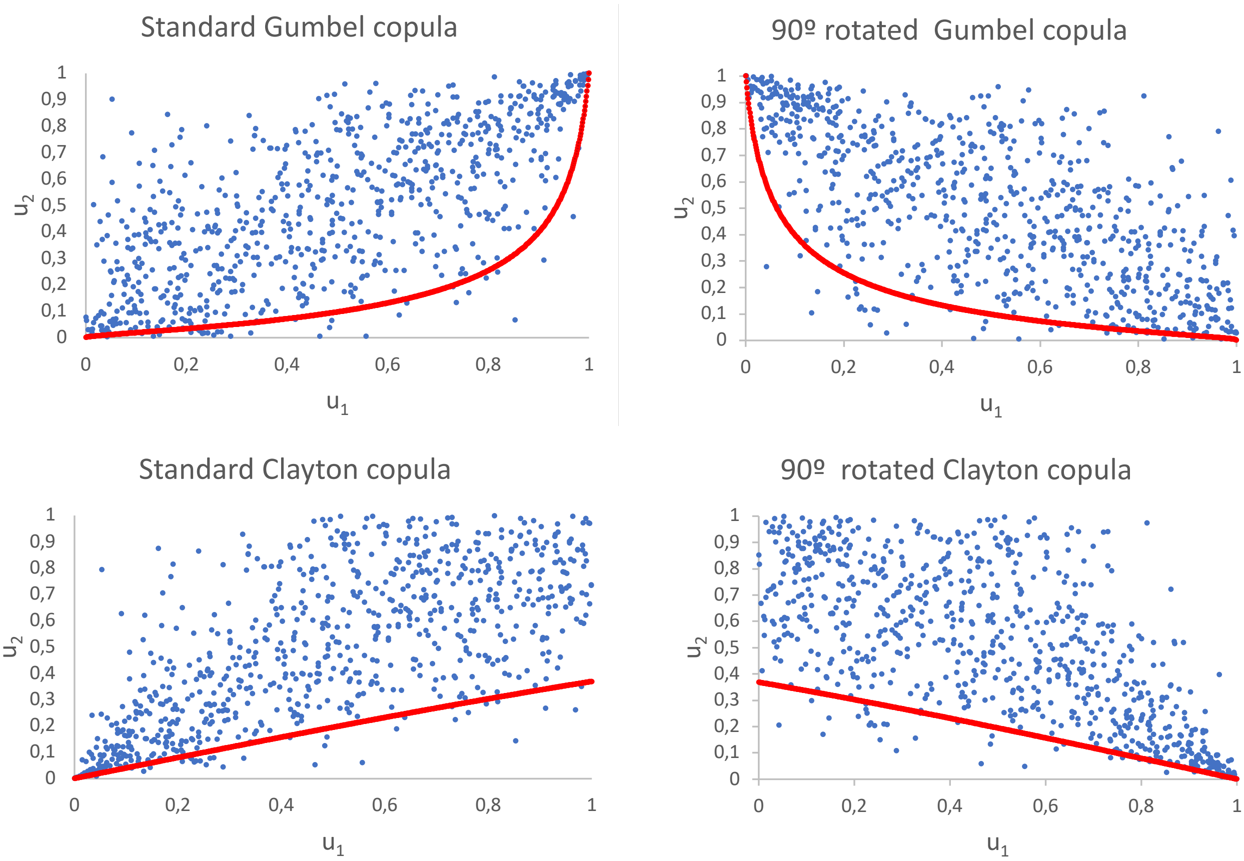
The non-elliptical copulas (Clayton, Gumbel and BB1 copulas) are rotated to allow for an asymmetric negative association. Graph 1 shows how the realisations from a simulation process change when the copula is rotated 90 degrees. The rotation of copulas provides a negative dependence between stock market and oil returns, which is not allowed by standard copulas. This sign change in the relationship between these variables has already been identified by the literature (see, for example: Boldanov, Degiannakis & Filis, 2016; Filis, Degiannakis & Floros, 2011)[23]. Appendix A presents the functional form of the copula employed in this article to get the main

22 Ao, S.I., Kim, H.K. & Amouzegar, M.A. (2017). *Transactions on Engineering Technologies: World Congress on Engineering and Computer Science 2015.* Springer; Jiang, C. (2012). *Does tail dependence make a difference in the estimation of systemic risk.* Boston College, Technical report, CoVaR and MES Working Paper; Joe, H. & Hu, T. (1996). "Multivariate distributions from mixtures of max-infinitely divisible distributions". *Journal of multivariate analysis*, Vol. 57, No. 2, pp. 240-265; Fischer, M.J. (2003). *Tailoring copula-based multivariate generalized hyperbolic secant distributions to financial return data: An empirical investigation.* Friedrich-Alexander-Universität Erlangen-Nürnberg, Lehrstuhl für Statistik und Ökonometrie, Technical report/Diskussionspapiere; Joe, H. (1997). *Multivariate models and multivariate dependence concepts*. Chapman & Hall/CRC.

23 Boldanov, R., Degiannakis, S. & Filis, G. (2016). "Time-varying correlation between oil and stock market volatilities: Evidence from oil-importing and oil-exporting countries". *International Review of Financial Analysis*, Vol. 48, No. C, pp. 209-220; Filis, G., Degiannakis, S. & Floros, C. (2011). "Dynamic correlation between stock market and oil prices: The case of oil-importing and oil-exporting countries". *International Review of Financial Analysis*, Vol. 20, No. 3, pp. 152-164.

results, based on analytical [24] and graphical tools[25] for the copula selection. Further details about functional form of the copulas may be found in Joe (2014)[26].

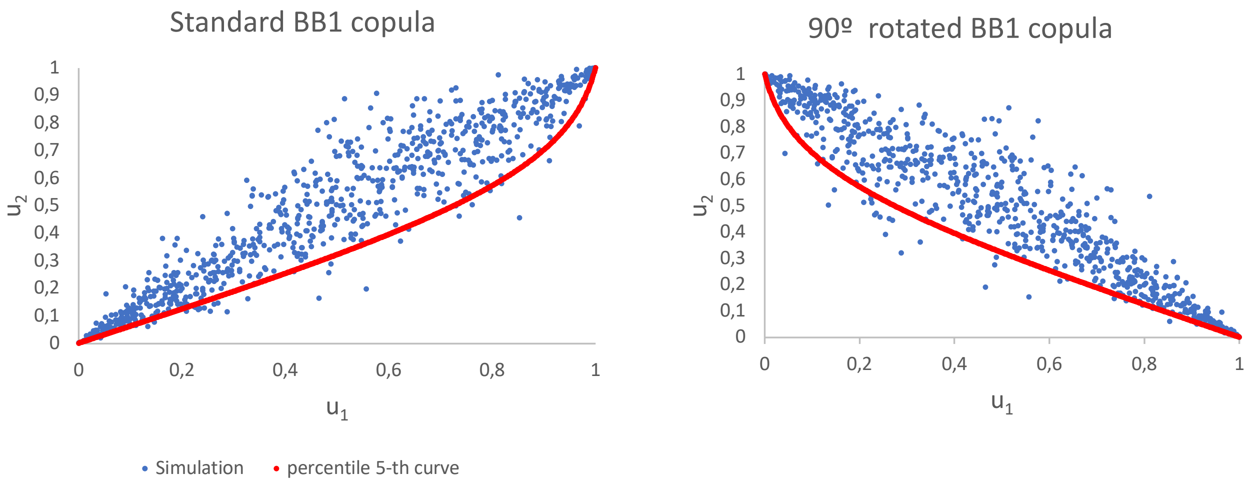
Simulation of two uniform variables with dependence defined by a copula GRAPH 1



24 The Akaike information criterion with correction for small sample sizes (AICc) is the main indicator for the selection of copulas in the literature. See, among others: Brechmann, E. & Schepsmeier, U. (2013). "CDVine: Modelling dependence with C-and D-vine copulas in R". *Journal of Statistical Software*, Vol. 52, No. 3, pp. 1-27; Reboredo, J.C. & Ugolini, A. (2015a). "A vine-copula conditional value-at-risk approach to systemic sovereign debt risk for the financial sector". *The North American Journal of Economics and Finance*, Vol. 32, No. C, pp. 98-123; Reboredo, J.C. & Ugolini, A. (2015b). "Downside/upside price spillovers between precious metals: A vine copula approach". *The North American Journal of Economics and Finance*, Vol. 34, No. C, pp. 84-102; Reboredo & Ugolini (2016), *op. cit.*; Rodriguez, J.C. (2007). "Measuring financial contagion: A copula approach". *Journal of Empirical Finance*, Vol. 14, No. 3, pp. 401-423; Reboredo, J.C. (2011). "How do crude oil prices co-move?: A copula approach". *Energy Economics*, Vol. 33, No. 5, pp. 948-955.

25 The empirical density functions, the lambda functions —Genest, C. & Rivest, L.P. (1993). "Statistical inference procedures for bivariate Archimedean copulas". *Journal of the American Statistical Association*, Vol. 88, No. 423, pp. 1.034-1.043; Aas, K., Czado, C., Frigessi, A. & Bakken, H. (2009). "Pair-copula constructions of multiple dependence". *Insurance: Mathematics and Economics*, Vol. 44, No. 2, pp. 182-198; Brechmann & Schepsmeier (2013), *op. cit.*; Schepsmeier, U. (2010). *Maximum likelihood estimation of c-vine pair-copula constructions on bivariate copulas from different families*. Center of Mathematical Sciences, Munich University of Technology— and tail concentration functions (TCF) —Pappadà, R., Durante, F. & Torelli, N. (2018). "A graphical tool for copula selection based on tail dependence". *Classification, (Big) Data Analysis and Statistical Learning*, Springer, pp. 211-218— are the main graphical tools used in the data analysis.

26 Joe, H. (2014). *Dependence Modeling with Copulas*. Chapman & Hall/CRC.



Source: Produced by the author. Note: The left figures show realisations of two uniform variables obtained from Gumbel, Clayton and BB1 copulas, while the figures on the right chart show the realisations obtained from the 90-degree rotation of these copulas. It can be seen a concentration of data in the upper right corner of the Gumbel copula, indicating upper tail dependence. The Clayton copula generates realisations where the concentration is greater at the lower left corner, reflecting the existence of lower tail dependence.

The red line shows the threshold below which, given the value taken by the variable u_1, the variable u_2 has 5% of the observations left below *i.e.* the 5% quantile of the variable e $u_1|u_2$.

800 realisations were generated in the simulation where the Clayton and Gumbel copulas have $\theta = 2$ and the BB1 has $\theta = 2$ and $\delta = 2{,}5$ as the parameter values.

The Markov switching specification is combined with the copula, so that the existing dependence is subordinated to the current state. In other words, following Equation (3), the joint distribution of oil derivative and variable income performances would be:

$$F(r_o, r_m; s_t) = C(F_o(r_o), F_m(r_m); s_t),$$

where s_t is the current regime or state at time t. We consider two regimes that allow us to generate a parsimonious and flexible model with an economic interpretation for each state. The joint distribution function will be the sum of the joint distribution under each state, weighted by the probability of being in each state at time t, $(P(s_t|I_{t-1})$ where I_{t-1} is the set of information up to the time *t-1*. To put it another way, we could see the joint distribution as a combination of copulas, where the weights of this combination vary over time depending on the probability of being in each state. The probability of switching from one state to another at t+1 conditioned to the current state follows a first order Markov chain[27]. At each moment t the likelihood of each observation is written as:

$$L_t(r_{o,t}, r_{m,t}; I_{t-1}, \Theta_t) = f(r_{o,t}, r_{m,t}|\Theta_{s_t=1}, I_{t-1})\, P(s_t = 1|I_{t-1}) + f(r_{o,t}, r_{m,t}|\Theta_{s_t=2}, I_{t-1})\, P(s_t = 2|I_{t-1}) \quad (7)$$

where $f(r_{o,t}, r_{m,t}|\Theta_{s_t=k}, I_{t-1}) = f(r_{o,t}) f(r_{m,t}) c(F(r_{o,t}), F(r_{m,t}); \Theta_{s_t=k})$ for $k = 1, 2$ and $\Theta_{s_t=k}$ are the copula parameters in state k, while the parameters of the marginal

27 See Fernández *et al.* (2012), *op. cit.*, to obtain a more detailed analysis regarding Markov Switching methodology and its application in econometrics.

distributions have been omitted for notational convenience. Using a maximum likelihood estimation means maximising the function $\sum_{t=1}^{T} \log(L_t)$, where L_t is defined by Equation (7). It should be noted that this function must be maximised by using non-linear methods, since it depends in a non-linear way on the set of parameters[28].

Data

The empirical exercise performed in this study uses weekly data for oil, exchange rates stock market during the period between 07/01/2000 and 23/10/2015. The sample includes several crises, where the oil price has experienced large oscillations[29].

Regarding oil price, the Brent spot price is employed sourced from the Energy Information Agency of the USA. [30] This is the key benchmark to set the price for refined oil, and it is considered a better approximation of market oil prices than OPEC prices (Sukcharoen *et al.*, 2014)[31]. Brent crude is traded in US dollars, hence it is transformed into euros using the USDEUR exchange rate available at the European Central Bank Statistical Data Warehouse[32]. Concerning the stock market variables, the Eurostoxx index is employed as a general index for the European stock market. We also consider its decomposition into ten subsectors taking into account the industrial classification benchmark (ICB) structure provided by Datastream. The subsectors are: oil and gas, basic materials, industrials, consumer goods, health care, consumer services, telecoms, utilities, financials and technology.

Graph 2 shows the correlation between the weekly oil returns and stock returns over time, using a rolling windows approach, with a five-year window length. The blue line refers to the Eurostoxx; the red line to basic materials; and the yellow line, to the health care sector. Two set of evidence are drawn from this graph: first the health sector shows a lower correlation with oil, compared with other stock returns; second, during period 2008-2009 the correlation between variables experiences a sharp increase, in line with the existence of a structural change pointed by the literature.

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28 The MATLAB function *fminsearch* offers good estimates, performing transformations of the parameters to keep them within a feasible region. The re-parameterisation procedure followed is similar to the one employed by Hamilton and Susmel(1994) to estimate their SWARCH model Hamilton, J. & Susmel, R. (1994). "Autoregressive conditional heteroskedasticity and changes in regime". *Journal of Econometrics*, Vol. 64, No.s 1-2, pp. 307-333.

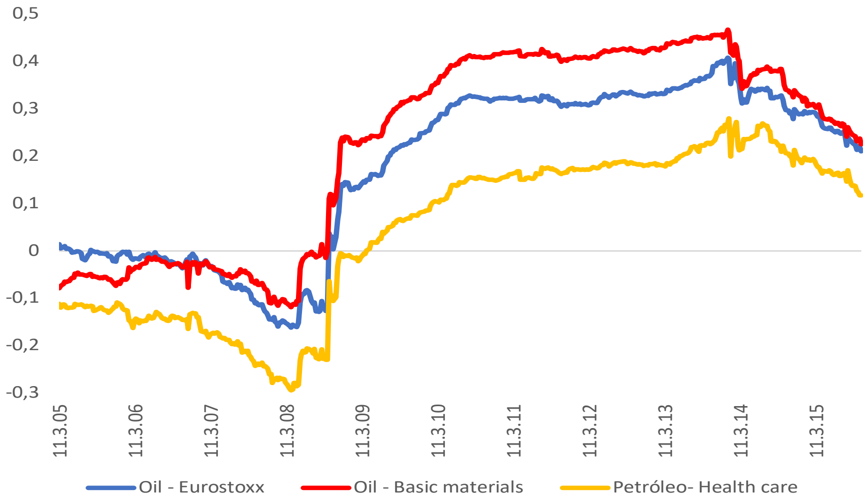
29 Examples of these fluctuations within the sample would include the "dot.com" crisis, the 2008 financial crisis, and the European sovereign debt crisis.

30 http://www.eia.doe.gov

31 Sukcharoen *et al.* (2014), *op. cit.*

32 https://sdw.ecb.europa.eu

Time-varying correlation over time between stock market returns and oil returns GRAPH 2



Source: Produced by the author. This figure shows the evolution of the correlation between oil and Eurostoxx (blue line), basic materials (red line) and health care (yellow line). The correlation evolves over time, following a rolling window approach using weekly returns and a 5-year window length.

Empirical exercise

This section presents in a first stage the results of the likelihood ratio test to justify the need to model a structural change in the dependence between variables. Then, the probability of being in each state of the Markov switching model are shown. Finally, the results of the stress test in terms of changes in the estimation of the Value at Risk (*VaR*) close this section.

3.4 Likelihood ratio

Table 2 shows the results of the likelihood ratio test. This test allows us to check the existence of a structural change by comparing the likelihood of two models, where one does not allow for changes in dependence (restricted model), and an extension where this change could occur. The null hypothesis of the test indicates that both models are equally likely, while the alternative hypothesis indicates the greater likelihood of the non-restricted model over the restricted one[33]. The null hypothesis is rejected at 5% for all sectors, justifying the need to model a structural change in the data, in line with the results obtained by other studies, such as Reboredo & Ugolini (2016)[34].

33 The likelihood ratio test requires some nuances. The restricted model includes a set of parameters that are not identified, such as the parameters of the Markov chain, which means that we do not fulfil the regularity conditions that allow us to approximate the distribution of the statistic under the null hypothesis to a chi square distribution. Hence, in order to obtain the distribution of the statistic according to the null hypothesis, we employ a *bootstrapping* procedure as the one used by Cai, J. (1994). "A Markov model of switching-regime ARCH". *Journal of Business & Economic Statistics*, Vol. 12, No. 3, pp. 309-316.

34 Reboredo & Ugolini (2016), *op. cit*.

Likelihood ratio statistical values and p-value obtained by means of a bootstrapping procedure TABLE 2

	A	B	C	D	E	F	G	H	I	J	K
LR	13.70	15.96	15.13	7.27	15.57	10.34	16.95	16.29	9.33	17.34	18.09
p-value	0.002	0.000	0.000	0.016	0.002	0.018	0.000	0.004	0.012	0.008	0.000

Source: Produced by the author. Note: This table shows the value of the likelihood ratio statistic, *i.e.* $LR = -2(\log(L_R) - \log(L_{UR}))$, where L_R is the maximum likelihood value of the restricted model (with no change between states) and L_{UR} is the maximum likelihood value of the unrestricted model (with change between states). Distribution under the null hypothesis is obtained through Monte Carlo simulation. The copulas under the restricted and unrestricted models are those which are optimal according to the Akaike criterion with correction for small samples.
A: Eurostoxx; B: oil and gas; C: basic materials; D: industrial; E: consumer goods; F: health care; G: consumer services; H: telecoms; I: utilities; J: financials; K: technology.

3.5 States of the Markov switching model

Graph 3 shows the probability of being in a scenario where oil and Eurostoxx returns are negatively related, with greater dependence when the oil price drops than when it rises. This scenario corresponds mainly to the period before 2008. After 2008, the dependence between sectors becomes positive, with lower tail dependence. This result of an increase in lower tail dependence coincides with the findings obtained by Reboredo & Ugolini (2016), Wen *et al.* (2012) & Aloui *et al.* (2013)[35]. Aloui *et al.* (2013) links the presence of lower tail dependence to the existence of herd behaviour among investors.

The following subsection analyses the response of the *VaR* of Eurostoxx and the health care sector when a upward or downward movement in oil prices materialise. These two sectors of the stock market were selected because of the different response shown to the same oil-related scenario (Table 3).

35 Reboredo & Ugolini (2016), *op. cit.*; Wen *et al.* (2012), *op. cit.*; Aloui *et al.* (2013), *op. cit*.

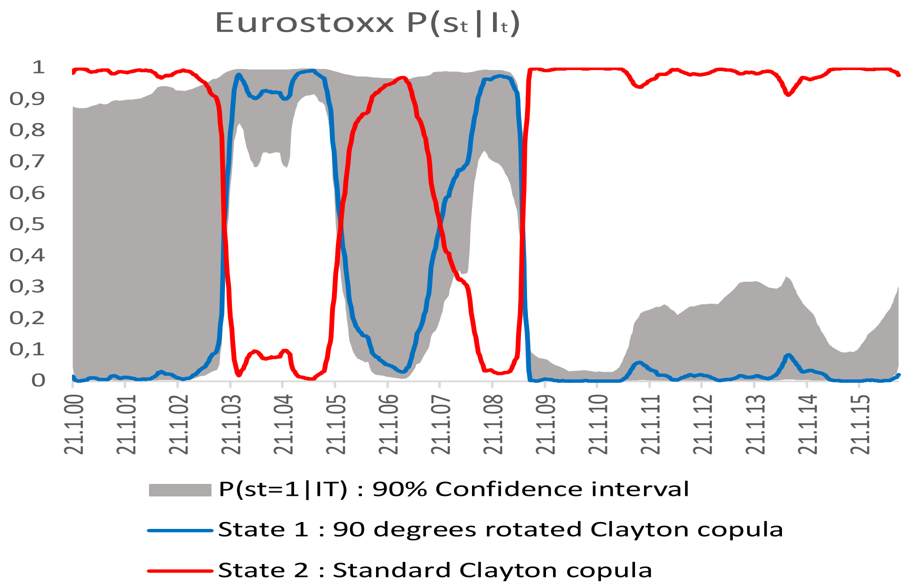
Estimated parameters of the joint distribution using a mixture of copulas where weights are given by the probability of being in each state

TABLE 3

	A	L		F	L
$\phi_{\kappa,0}$	0,00	0,00	$\phi_{\kappa,0}$	0,00**	0,00
	(0,00)	(0,00)		(0,00)	(0,00)
$\phi_{\kappa,1}$	-0,07**	0,03	$\phi_{\kappa,1}$	-0,10***	0,04
	-0,04	-0,04		(0,04)	(0,04)
ω_{κ}	0,00**	0,00*	ω_{κ}	0,00	0,00*
	(0,00)	(0,00)		(0,00)	(0,00)
α_{κ}	0,00	0,05**	α_{κ}	0,04*	0,04**
	(0,04)	(0,03)		(0,03)	(0,03)
β_{κ}	0,84***	0,90***	β_{κ}	0,88***	0,90***
	-0,07	-0,03		(0,07)	(0,03)
γ_{κ}	0,22***	0,08**	γ_{κ}	0,06	0,07**
	(0,07)	(0,04)		(0,05)	(0,04)
λ_{κ}	-0,37***	-0,27***	λ_{κ}	-0,21***	-0,23***
	(0,04)	(0,05)		(0,05)	(0,05)
η_{κ}	12,13***	11,99***	η_{κ}	8,77***	13,98***
	(0,46)	(0,89)		(0,86)	(0,89)
	90RClayton-Clayton			Student-Gaussian	
$\theta_{st=1}$	0,3877***		$\rho_{st=1}$	-0,2788***	
	(0,16)			(0,09)	
$\theta_{st=2}$	0,2742***		$\eta_{st=1}$	6,7021***	
	(0,06)			(1,54)	
ρ_{11}	0,9838***		$\rho_{st=2}$	0,1279**	
	(0,01)			(0,06)	
ρ_{22}	0,9956***		ρ_{11}	0,9793***	
	(0,00)			(0,01)	
LL	3104,40		ρ_{22}	0,9890***	
				(0,01)	
			LL	3086,64	

This table shows the estimators and their standard deviations (in brackets) for the parameters of the marginal model given by Equations (4), (5) and (6), and for the best mixture of copulas following the *AICc information criterion* ***/**/* indicates statistical significance at 1/5/10%.
The copula evolves following a two-state *Markov switching* specification. ρ_{ii} and indicates the probability of remaining in state i. Each pair of columns x-L reveals a fully estimated model between the stock market (x) and oil returns (L: oil): A: EUROSTOXX; F: health care. The other sectors are available under request.
The EUROSTOXX-oil link is defined under scenario 1 by a Clayton copula rotated 90 degrees, while under state 2 a standard Clayton copula is used, *i.e.* without rotation. The relationship between the health care sector and oil is defined under state 1 by a Student t copula, while under state 2 a Gaussian copula is used.
Source: Produced by the author.

Smoothed probabilities of being in each state GRAPH 3



Source: Produced by the author. Note: The graph shows the smoothed probabilities, *i.e.* the probabilities of being under one state or another, given all the information contained in the sample. These smoothed probabilities are obtained from Kim's algorithm (1994)[36]. The grey area shows a confidence interval of 90% obtained by bootstrapping. The smoothed probabilities for the other sectors are available if requested[37].

3.6 Change in the Value-at-Risk of the stock market returns when an oil-related scenario materialises

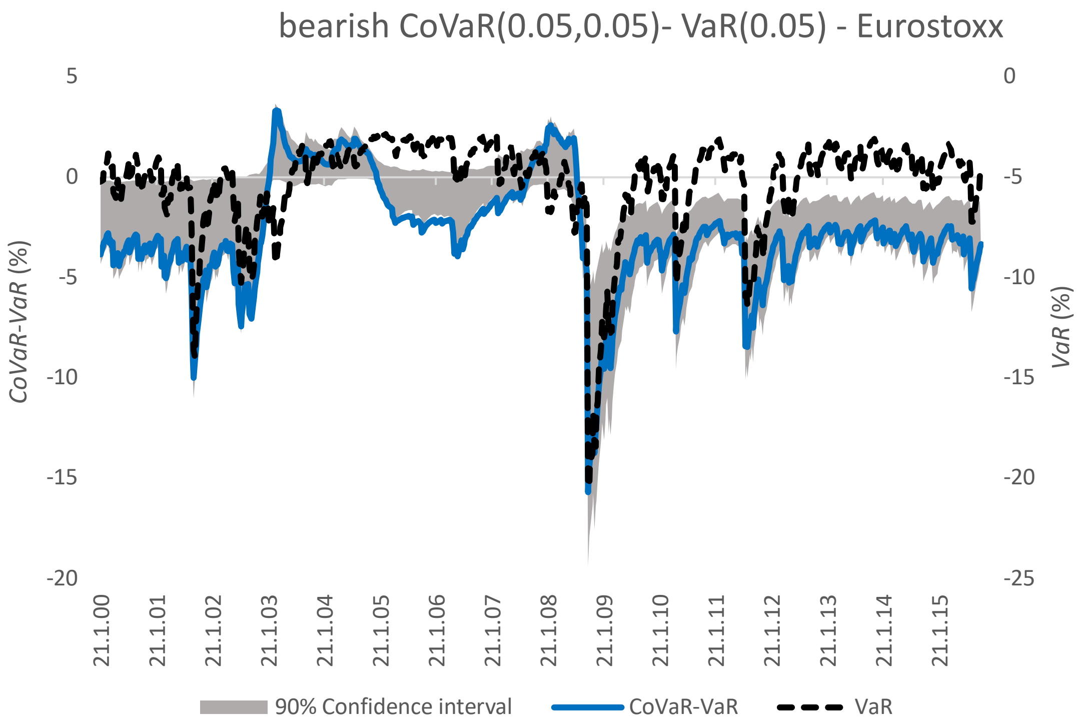
Graphs 4 and 5 show the main changes in the *VaR* when an oil-related scenario materialises. The left axis shows the difference between the *CoVaR* and the *VaR* (blue line), and the grey area indicates the 90% confidence interval. The right axis presents the unconditional Value-at-Risk (black line). The Eurostoxx *VaR* is around -6%, while losses in the health care sector are lower, with returns around -5%. Likewise, under the bearish oil-related scenario, this change in losses indicates lower variability in health care sector returns (between 2% and -2%) than in the Eurostoxx (between 5% and -12%). On the other hand, when a bullish oil-related scenario materialises, the change in the *VaR* of the health care sector presents a different pattern from what Eurostoxx shows. This difference may lead to diversification benefits because of the different response shown by the two stock assets to the same oil-related scenario.

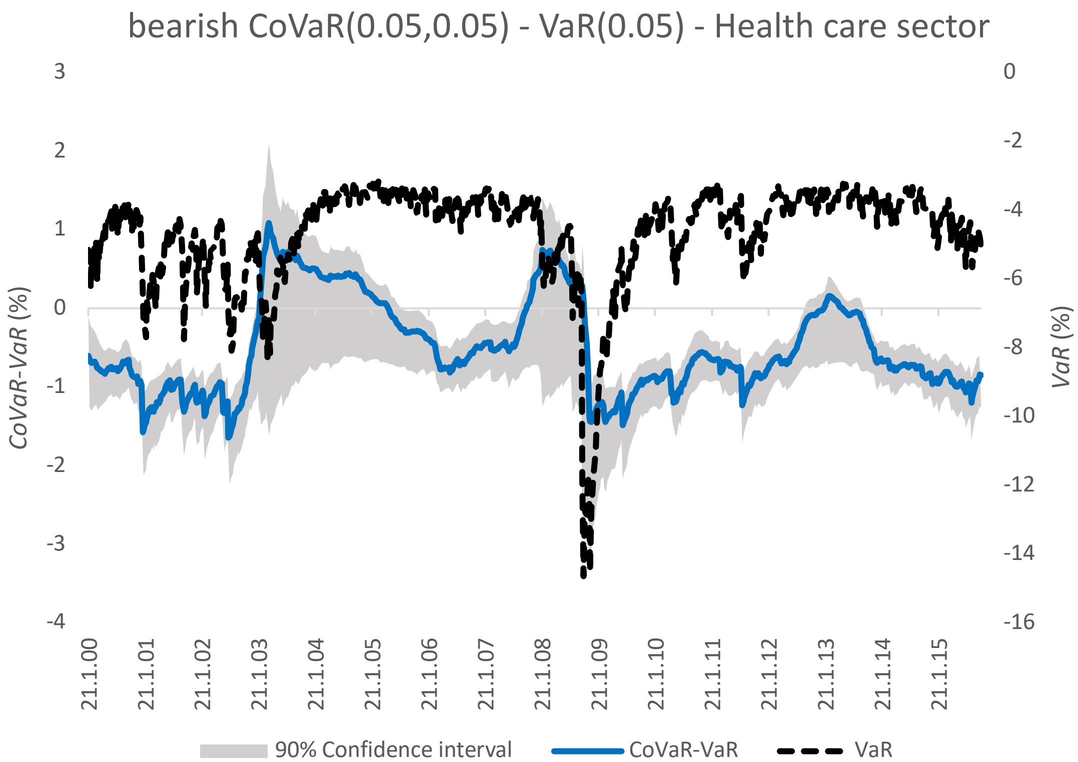
36 Kim, C. (1994). "Dynamic linear models with Markov-switching". *Journal of Econometrics,* Vol. 60, No.s 1-2, pp. 1-22.

37 They may also be consulted in Ojea Ferreiro, Javier. (2019). Structural change in the link between oil and the European stock market: implications for risk management. Dependence Modeling. 7. 53-125.

Change in the *VaR* of stock market returns when a bearish oil-related scenario materialises

GRAPH 4

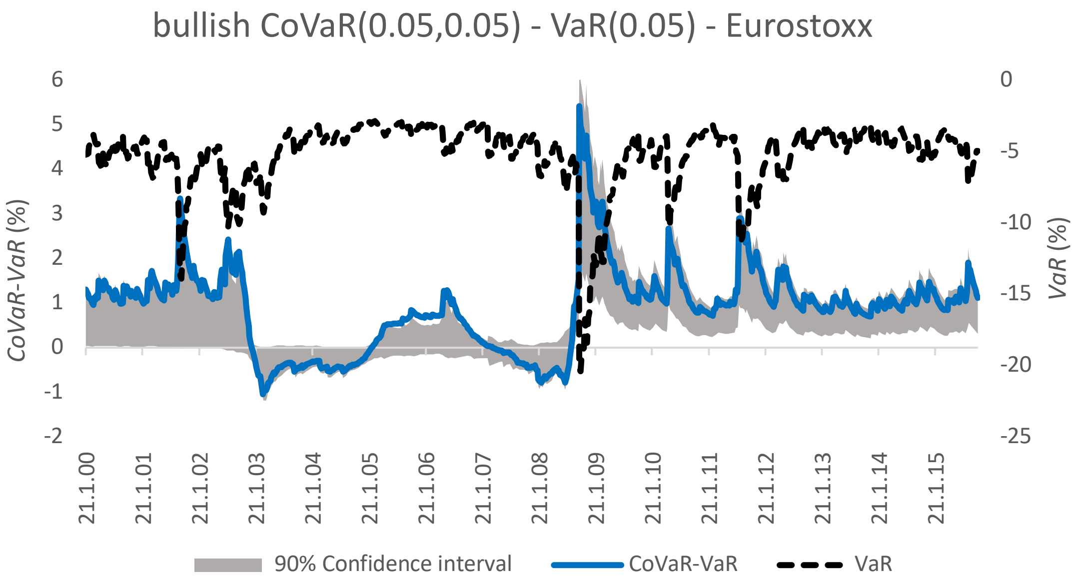


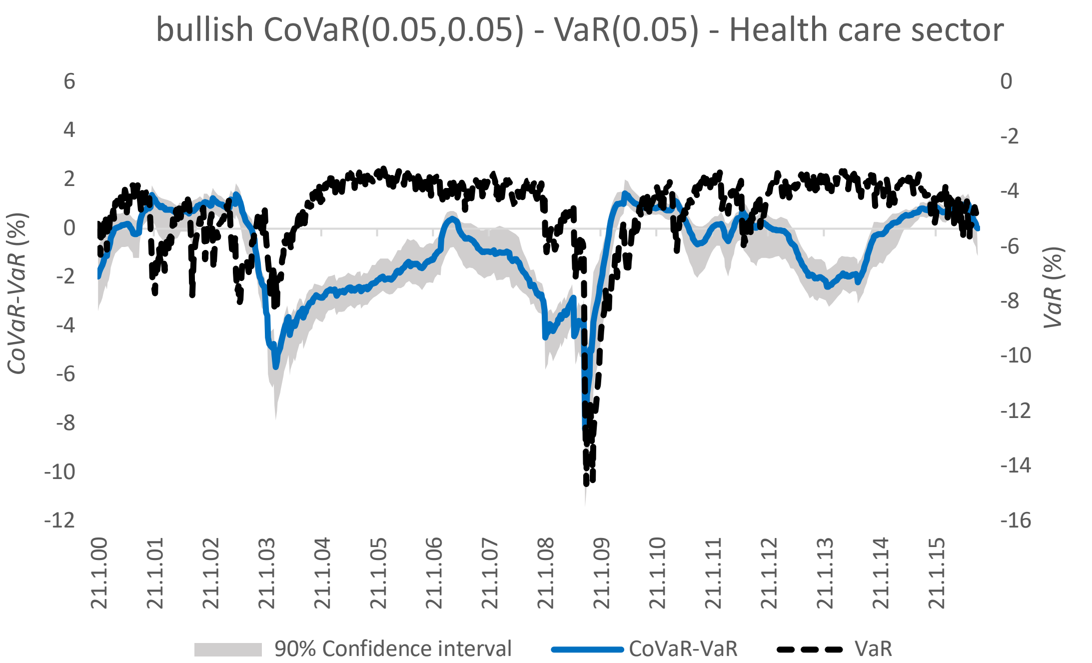


Source: Produced by the author.

Change in the *VaR* of stock market returns when a bullish oil-related scenario materialises

GRAPH 5





Source: Produced by the author.

4 Conclusions

This article proposes the use of a copula methodology combined with a stochastic changing regime approach to design stress scenarios related to energy events. This framework allows us to analyse the response of the stock market to energy-related scenarios. This methodology provides a flexible way to identify potential changes in the risk measure employed by investors as benchmark for management purposes, *i.e.* the Value-at-Risk (*VaR*).

The use of the Conditional Value-at-Risk, or *CoVaR* provides information about the connection between both assets under extreme scenarios. In other words, it gives a more robust estimation for outliers than mean responses. In econometric terms, the proposal model reflects non-linearities, asymmetric behaviour and tail dependence, along with structural changes in the relationship between oil and stock returns.

The empirical exercise uses weekly data from European stock markets and the oil price for the period 2000-2015, finding a structural change in the relationship between variables upon the outbreak of the financial crisis in 2008. This change may be found to be closely related to the economic cycle. Before 2008, the relationship between both sectors was negative: when the oil price fell, the drop was reflected in higher returns on the European stock markets. This link could be explained by the increased profit margin, consequence of the drop in the firms' production costs. An increases in oil prices would be reflected in the sales price, leading to an inflationary process in the medium term. This is a gradual process, which would explain the lower dependence when the oil price increases.

The 2008 crisis trigged losses in European firms decreasing their oil demand, which in turn prompted a fall in the price. This relationship through the aggregate demand would explain the positive dependence in the lower tail after 2008. This relationship between the co-movement of economic sectors and oil price through the economic cycle has already been pointed out by Andreopoulos (2009)[38] for the US economy. The results of the stress test showed different responses across sectors to the same oil-related scenario. The difference in the response could be explained by the elasticity of the demand of these sectors' products and the sensitivity of their cash flows to changes in the oil price.

38 Andreopoulos, S. (2009). "Oil matters: Real input prices and US unemployment revisited". *The BE Journal of Macroeconomics*, Vol. 9, No. 1.

Further research should analyse the potential role of exchange rate to mitigate the negative effects of abrupt movements in oil prices on the stock market. Since oil prices are denominated in dollars that are then translated into euros, a transitory shock in oil prices can be alleviated by the right movement in the exchange rate. Additional robustness checks can be performed by slightly modifying the model. The results of this study are relevant for portfolio managers, who wish to reduce the oil exposure of their stock portfolios. Investors also have an interest in understanding possible diversification strategies in case of a materialisation of an extreme movement in oil prices. The regulatory authorities also have an interest in monitoring the behaviour of different productive sectors, within the context of the transition to green energy. Lastly, policy makers need quantitative measures that can translate the response of stock market to oil-price instabilities into potential losses.

5 Appendix

Appendix A: Functional forms of the copulas

Gaussian copula. This copula has a parameter ρ which indicates the linear correlation. There is no closed formula expression for the Gaussian copula since it is an implicit copula.

The copula density function is:

$$c(u_1, u_2; \rho) = \frac{1}{\sqrt{1-\rho^2}} exp\left\{-\frac{\rho^2\Phi^{-1}(u_1)^2 - 2\rho\Phi(u_1)\Phi(u_2) + \rho^2\Phi^{-1}(u_2)^2}{2(1-\rho^2)}\right\}$$

Where Φ^{-1} indicates the inverse function of the Gaussian distribution function. u_1 and u_2 are the integral representation of the marginal distributions function. Meyer (2013)[39] analyses this copula in depth.

Student t copula. This type of copula allows positive and negative tail dependence. The parameter ρ measures the correlationwhile the parameter η, gathers the numbers of degrees of freedom controlling the probability of observing extreme realisations of the variables analysed. As with the Gaussian copula, this is an implicit copula, and its functional form is therefore obtained through the double integral of its copula density.

The copula density function is:

$$c(u_1, u_2; \rho, \eta) = K \frac{1}{\sqrt{1-\rho^2}} exp\left\{-\frac{T_\eta^{-1}(u_1)^2 - 2\rho T_\eta^{-1}(u_1)T_\eta^{-1}(u_2) + T_\eta^{-1}(u_2)^2}{\eta(1-\rho^2)}\right\}^{\frac{\eta+2}{2}}$$

$$\left[\left(1+\eta^{-1}T_\eta^{-1}(u_1)^2\right)\left(1+\eta^{-1}T_\eta^{-1}(u_2)^2\right)\right]^{\frac{(\eta+1)}{2}}$$

where $= \; K = \Gamma\left(\frac{\eta}{2}\right)\Gamma\left(\frac{(\eta+1)}{2}\right)\Gamma\left(\frac{(\eta+2)}{2}\right)$

Demarta & McNeil (2005)[40] present a detailed study about this copula.

39 Meyer, C. (2013). "The bivariate normal copula". *Communications in Statistics-Theory and Methods*, Vol. 42, No. 13, pp. 2.402-2.422.

40 Demarta, S. & McNeil, A.J. (2005). "The t copula and related copulas". *International Statistical Review/Revue Internationale de Statistique*, Vol. 73, No. 1, pp. 111-129.

Clayton copula. This copula allows a positive relationship between variables with lower tail dependence. The parameter θ of the copula is between 0 (independence) and infinity (perfect dependence). The functional form of the Clayton copula is

$$C(u_1, u_2; \theta) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-\frac{1}{\theta}},$$

and the density copula is:

$$c(u_1, u_2; \theta) = (\theta + 1)\left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-2-\frac{1}{\theta}}(u_1 u_2)^{-\theta-1}.$$

The Clayton copula rotated 90 degrees is obtained as:

$$C_{90}(u_1, u_2; \theta) = u_2 - C(1 - u_1, u_2),$$

where C (...) corresponds to the Clayton copula, while its copula density is $c(1 - u_1, u_2; \theta)$, where the copula density used is that of Clayton.

Gumbel copula. This copula allows a positive relationship between variables with upper tail dependence. The parameter θ of the copula is between 1 (independence) and infinity (perfect dependence). The functional form of the Gumbel copula is:

$$C(u_1, u_2; \theta) = \exp\left(-\{(-\log(u_1))^{\theta} + (-\log(u_2))^{\theta}\}^{\frac{1}{\theta}}\right),$$

and the density copula is:

$$c(u_1, u_2; \theta) = (A + \theta - 1)A^{1-2\theta} \exp(-A)\,(u_1 u_2)^{-1}\left(-log(u_1)\right)^{\theta-1}\left(-log(u_2)\right)^{\theta-1}.$$

Where $A = \left[\left(-log(u_1)\right)^{\theta} + \left(-log(u_2)\right)^{\theta}\right]^{\frac{1}{\theta}}$.

The Gumbel copula rotated 90 degrees is obtained as:

$$C_{90}(u_1, u_2; \theta) = u_2 - C(1 - u_1, u_2),$$

where C(...) corresponds to the Gumbel copula, while its copula density is $c(1 - u_1, u_2; \theta)$, where the copula density used is that of Gumbel.

BB1 copula or Clayton-Gumbel copula. This copula allows a positive relationship between variables with asymmetric dependence in both tails. The set of parameters θ and δ of the copula modules the type of dependence. The parameter θ adopts values between 0 and infinity, regulating lower tail dependence, while the parameter δ adopts values between 1 and infinity, modelling the relationship in the upper tail.[41] The functional form of the Clayton-Gumbel copula is:

41 See for example Venter, G. (2002) Tails of copulas. In Proceedings of the Casualty Actuarial Society, volume 89, pages 68-113 or Nicoloutsopoulos, D. (2005) Parametric and Bayesian non-parametric estimation of copulas. PhD thesis, University of London

$$C(u_1, u_2; \theta) = \left(1 + \left[\left(u_1^{-\theta} - 1\right)^{\delta} + \left(u_2^{-\theta} - 1\right)^{\delta}\right]^{\frac{1}{\delta}}\right)^{-\frac{1}{\theta}},$$

and the density copula is:

$$c(u_1, u_2; \theta) = (u_1 u_2)^{-\theta-1}(ab)^{\delta-1}c^{\frac{1}{\delta}-2}d^{-\frac{1}{\theta}-1}\left\{d^{-1}c^{\frac{1}{\delta}}(1+\theta) + \theta(\delta-1)\right\}.$$

Where $a = u_1^{-\theta} - 1, b = u_2^{-\theta} - 1, c = a^{\delta} + b^{\delta}\ y\ d = 1 + c^{\frac{1}{\delta}}.$

The BB1 copula rotated 90 degrees is obtained as:

$$C_{90}(u_1, u_2; \theta) = u_2 - C(1 - u_1, u_2),$$

where *C* (…) corresponds to the BB1 copula, while its copula density is $c(1 - u_1, u_2; \theta)$, where the copula density used is that of BB1.